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DEPARTMENT OF ECONOMICS

RESEARCH MEMORANDUM

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COORDINATE-FREE INTERPRETATIONS OF THE OPTIMAL COSTS FOR LQ-PROBLEMS SUBJECT TO IMPLICIT SYSTEMS

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Control Systems

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LINEAR-QUADRATIC CONTROL WITH AND WITHOUT STABILITY SUBJECT TO GENERAL IMPLICIT CONTINUOUS-TIME SYSTEMS: COORDINATE-FREE INTERPRETATIONS OF THE OPTIMAL COSTS IN TERMS OF DISSIPATION INEQUALITY AND LINEAR MATRIX INEQUALITY; EXISTENCE AND UNIQUENESS OF OPTIMAL CONTROLS AND STATE TRAJECTORIES

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ABSTRACT

We consider linear-quadratic control problems with and without stability, subject to an arbitrary implicit continuous-time system, in a simple *distributional* framework, and it will be shown that the associated optimal costs, if existent, are solutions of our *Dissipation Inequality* for implicit systems. This concept is related to the *Linear Matrix Inequality*, which is expressed in *original* system coefficients only, and the above-mentioned optimal costs turn out to be characterizable *uniquely* by certain solutions of this inequality. However, these solutions need *not* be *rank minimizing* if the underlying system is *not* standard, and we will specify why this is the case. Our statements are valid for regular as well as for singular problems, and the possible significance of the algebraic Riccati equation will be illustrated for both regular *and* singular problems. Furthermore, we will present necessary and sufficient conditions for *solvability* of our problems and for *existence* of optimal controls and associated optimal state trajectories. Finally, we will elaborate on the *uniqueness* of these controls and state trajectories.

KEYWORDS

Linear-quadratic control, implicit system, impulsive-smooth distributions, regularity and singularity, weakly unobservable and strongly controllable subspaces, dissipation inequality, linear matrix inequality, rank minimizing solutions, algebraic Riccati equation, output and state stabilizability, left invertibility in the strong sense.

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1. Introduction.

This paper is concerned with the concepts of *Dissipation Inequality* (DI) and *Linear Matrix Inequality* (LMI) for general *implicit* continuous-time linear systems with constant coefficients. In particular, we will investigate the strong relation between these concepts and Linear-Quadratic Control Problems (LQCPs) subject to implicit systems.

To the best of the author's knowledge, these issues have been investigated in depth for *standard* systems only. For standard systems the main points are as follows.

Consider the standard system:

$$\bar{x}'(t) = \bar{A}\bar{x}(t) + \bar{B}\bar{u}(t), \tag{1.1a}$$

$$\bar{y}(t) = \bar{C}\bar{x}(t) + \bar{D}\bar{u}(t), \tag{1.1b}$$

with $\bar{x}(0) = \bar{x}_0 \in \mathbb{R}^n$, $\bar{u}(t) \in \mathbb{R}^m$, $\bar{x}(t) \in \mathbb{R}^n$, $\bar{y}(t) \in \mathbb{R}^r$ for all $t \geq 0$, and all matrices involved are real-valued and constant. In addition, we define the objective function

$$\bar{J}(\bar{x}_0, \bar{u}) := {}_0\!\int^{\infty} \bar{y}'(t)\bar{y}(t)dt, \tag{1.2}$$

where $\bar{u} \in \mathcal{L}^m_{2,\ \mathrm{loc}}(\mathbb{R}^+)$, the m-vector version of $\mathcal{L}_{2,\ \mathrm{loc}}(\mathbb{R}^+)$, the space of locally square-integrable functions over $\mathbb{R}^+ = [0, \infty)$. Then for every $\bar{x}_0 \in \mathbb{R}^n$ we can introduce the functions

$$\bar{J}^-(\bar{x}_0) := \inf\{\bar{J}(\bar{x}_0, \bar{u}) \,|\, \bar{u} \in \mathcal{L}^m_{2,\ \mathrm{loc}}(\mathbb{R}^+)\}, \tag{1.3}$$

$$\bar{J}^+(\bar{x}_0) := \inf\{\bar{J}(\bar{x}_0, \bar{u}) \,|\, \bar{u} \in \mathcal{L}^m_{2,\ \mathrm{loc}}(\mathbb{R}^+),\ \lim_{t\to\infty} \bar{x}(t) = 0\}, \tag{1.4}$$

as $0 \leq \bar{J}^-(\bar{x}_0) \leq \bar{J}^+(\bar{x}_0)$. In [1] these functions are called the optimal cost for the Linear-Quadratic Control Problem (LQCP) *without* and the optimal cost for the LQCP *with* stability, respectively. The LQCPs are called *regular* if $\ker(\bar{D}) = 0$ and *singular* if $\ker(\bar{D}) \neq 0$.

The optimal costs $\bar{J}^-: \mathbb{R}^n \to \mathbb{R}^+$ and $\bar{J}^+: \mathbb{R}^n \to \mathbb{R}^+$ satisfy the so-called *Dissipation Inequality* (DI) if, for every $\bar{x}_0 \in \mathbb{R}^n$, $\bar{J}^+(\bar{x}_0) < \infty$ [1] - [3]. A function $\bar{V}: \mathbb{R}^n \to \mathbb{R}$ is said to satisfy the DI if for every $\bar{x}_0 \in \mathbb{R}^n$, every $T \geq 0$ and every locally square-integrable function $\bar{u}: [0, T] \to \mathbb{R}^m$ it holds that

$$\int_0^T \bar{y}'(t)\bar{y}(t)dt + \bar{V}(\bar{x}(T)) \geq \bar{V}(\bar{x}_0), \tag{1.5}$$

and $\bar{V}(0) = 0$. Next, we define for any $\bar{K} \in \mathbb{R}^{n\times n}$ the dissipation matrix

$$\bar{F}(\bar{K}) := \begin{bmatrix} \bar{C}'\bar{C} + \bar{A}'\bar{K} + \bar{K}\bar{A} & \bar{K}\bar{B} + \bar{C}'\bar{D} \\ \bar{B}'\bar{K} + \bar{D}'\bar{C} & \bar{D}'\bar{D} \end{bmatrix} \tag{1.6}$$

[4], and $\bar{K}$ is said to satisfy the *Linear Matrix Inequality* (LMI) if $\bar{K}$ is symmetric and $\bar{F}(\bar{K}) \geq 0$. The set of real symmetric solutions of the LMI will be denoted by $\bar{\Gamma}$:

$$\bar{\Gamma} := \{\bar{K} \in \mathbb{R}^{n\times n} \mid \bar{K} = \bar{K}', \ \bar{F}(\bar{K}) \geq 0\}. \tag{1.7}$$

It is well known [1], [4] that there exists an integer $\bar{\rho} \geq 0$ such that, for every $\bar{K} \in \bar{\Gamma}$, rank $(\bar{F}(\bar{K})) \geq \bar{\rho}$. In fact, $\bar{\rho}$ = normal rank $(\bar{T}(s))$, with $\bar{T}(s) = \bar{D} + \bar{C}(sI - \bar{A})^{-1}\bar{B}$, the transfer matrix of (1.1). If $\bar{K} \in \bar{\Gamma}$ is such that rank $(\bar{F}(\bar{K})) = \bar{\rho}$, then $\bar{K}$ will be called *rank minimizing*. If $\ker(\bar{D}) = 0$, then $\bar{\rho} = m$ and $\bar{K} \in \bar{\Gamma}$ is rank minimizing if and only if $\bar{K} = \bar{K}' \in \mathbb{R}^{n\times n}$ satisfies the algebraic Riccati equation

$$\bar{C}'\bar{C} + \bar{A}'\bar{K} + \bar{K}\bar{A} - (\bar{K}\bar{B} + \bar{C}'\bar{D})(\bar{D}'\bar{D})^{-1}(\bar{B}'\bar{K} + \bar{D}'\bar{C}) = 0. \tag{1.8}$$

Proposition 1.1.

Assume that for all $\bar{x}_0 \in \mathbb{R}^n$, $\bar{J}^+(\bar{x}_0) < \infty$. Then there exist real symmetric matrices $\bar{K}^+ \in \bar{\Gamma}$ and $\bar{K}^- \in \bar{\Gamma}$, $\bar{K}^+ \geq \bar{K}^- \geq 0$, such that, for all $\bar{x}_0 \in \mathbb{R}^n$,

$$\bar{J}^-(\bar{x}_0) = \bar{x}_0'\bar{K}^-\bar{x}_0, \quad \bar{J}^+(\bar{x}_0) = \bar{x}_0'\bar{K}^+\bar{x}_0.$$

In addition, $\bar{K}^+$ and $\bar{K}^-$ are rank minimizing;

$\bar{K}^+ \geq \bar{K}$ for all $\bar{K} \in \bar{\Gamma}$;

$\bar{K}^- \leq \bar{K}$ if $\bar{K} \geq 0$ and $\bar{K} \in \bar{\Gamma}$ is rank minimizing.

Proof. The claims concerning existence of $\bar{K}^-$ and $\bar{K}^+$ can be found in [3] and [1] if (A, B) is controllable and in [6], [5] if (A, B) is stabilizable, and it is clear [6] that $\bar{J}^+(\bar{x}_0) < \infty$ for every $\bar{x}_0 \in \mathbb{R}^n$ if and only if (A, B) is stabilizable. The remaining statements on $\bar{K}^+$ are in [1] and [4], those on $\bar{K}^-$ are in [6] - [8].

Remark 1.2.

The characterizations of $\bar{K}^+$ and $\bar{K}^-$ in Proposition 1.1 can be given in words as follows. $\bar{K}^+$ is the *largest* solution of the LMI, $\bar{K}^-$ is the *smallest positive semi-definite rank minimizing* solution of the LMI. We establish that the optimal costs (1.3) - (1.4) can be characterized in terms of rank minimizing solutions of the LMI.

Optimal controls and resulting optimal state trajectories for regular LQCPs are ordinary functions [1], [6]. For singular LQCPs, however, optimal controls and/or resulting state trajectories are in general *distributions* [9], and singular LQCPs subject to standard systems are solved completely in terms of distributions in [10], [6], [8].

In the present paper we will investigate LQCPs without and with stability subject to the *implicit* generalization of (1.1):

$$Ex'(t) = Ax(t) + Bu(t), \tag{1.9a}$$

$$y(t) = Cx(t) + Du(t), \tag{1.9b}$$

where $E,\ A \in \mathbb{R}^{l\times n}$, $B \in \mathbb{R}^{l\times m}$, $C \in \mathbb{R}^{r\times n}$, $D \in \mathbb{R}^{r\times m}$, $e = \text{rank}\ (E)$, $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$ and $y(t) \in \mathbb{R}^r$ for all $t \geq 0$. Whereas for standard systems of the form (1.1a) every initial condition $x_0 = x(0)$ is *consistent* [11] in the sense that (1.1a) has a solution $x = x(x_0,\ u)$ with $x(0^+) = x_0$, there may be *inconsistent* points $x_0 \in \mathbb{R}^n$ if E is *singular*, i.e., if E is not invertible, then there may be points $x_0 \in \mathbb{R}^n$ for which (1.1a) does not have an ordinary (measurable) solution x with $x(0^+) = x_0$ [12].

Example 1.3 [13].

The implicit system

$$\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}$$

has only one solution, namely $x_1 = 0$, $x_2 = 0$. Hence $x_0 = 0$ is consistent and $x_0 \neq 0$ is inconsistent.

Various contributors on implicit systems make distinction between consistent and inconsistent points by interpreting $x_0 \in \mathbb{R}^n$ as the value of the state variable x of (1.9a) immediately *before* starting the dynamical process: $x_0 = x(0^-)$. Then a point $x(0^-) \in \mathbb{R}^n$ is considered consistent if there exists an ordinary solution x of (1.9a) with $x(0^-) = x(0^+)$. However, in electrical circuits $x_0 = x(0^-)$ may be *inconsistent*, for instance if the state value at $t = 0^-$ represents the potential of a capacitor, immediately before closing a switch. Then, an inconsistent value of $x(0^-)$ might give rise to an *impulsive* solution of the system if this switch is closed.

Example 1.3 (continued).

The system in Example 1.3 corresponds to a simple circuit with unit capacitor only, x_2 denoting its potential, x_1 the current. The switch is closed at $t = 0$. If $x_{01} := x_1(0^-) = 0$, but $x_{02} := x_2(0^-) \neq 0$, then the solution of Example 1.3 becomes [13] $x_2 = 0$, $x_1 = -\ x_{02}\delta(t)$, with $\delta(t)$ denoting the Dirac delta function.

Such Examples strongly suggest to reconsider (1.9a) in a distributional framework, so as to allow the implicit system to exhibit distributional behavior if $x_0 = x(0^-)$ is inconsistent. Observe, that such behavior may occur even if the input u is an ordinary function; in Example 1.3 there is no control possibility at all.

In [14] Cobb formulated an LQCP with stability for an implicit system (1.9) in terms of distributions. There, sE - A is assumed to be invertible, as a result of which the distributional version of (1.9a) has a unique (possibly distributional) solution for every pair (x_0, u), with u any (possibly distributional) input, and $x_0 = x(0^-) \in \mathbb{R}^n$ [14].

We will set up LQCPs subject to *any* implicit system (1.9) in a *distributional* framework in Section 2 (our Preliminaries).

Consequently, for certain choices of inconsistent points and chosen inputs our distributional implicit system equation may have more than one (distributional) solution, or even no solutions at all. This observation leads to extra difficulties in the formulation of our LQCPs: We must explicitly require that infimization be done over input *functions* and corresponding state *functions* (if any). In standard LQCPs this difficulty is no issue, as solutions of (1.1a) are automatically functions if the chosen inputs are, whereas in [14] $C = \begin{bmatrix} I \\ 0 \end{bmatrix}$, $D = \begin{bmatrix} 0 \\ I \end{bmatrix}$, and hence infimization of the objective function "forces" both inputs and state trajectories to be functions. Also in [15] existence of solutions for the implicit system equation is guaranteed by assuming that sE - A is invertible; yet, the matrices C and D are allowed to be more general than in [14].

Now it is our objective to (re)start the treatment of LQCPs subject to implicit systems (1.9) from unambiguous problem formulations rather than from unnecessarily restrictive assumptions.

Moreover, we will consider regular as well as singular LQCPs. We will call any LQCP subject to any system (1.9) *regular* if output *functions* are generated by control *functions* and state *functions* only, and we will call such a problem *singular* if this system property does not occur [16]. Whereas LQCPs subject to standard systems are regular if and only if the input weighting matrix in (1.2) is positive definite, LQCPs subject to implicit systems may be regular if $\ker(D) \neq 0$, and they may be singular if $\ker(D) = 0$. For examples of such problems, see [16, Section 1]. It is proven in [16] that regularity of any LQCP subject to any system (1.9) can be characterized by a condition on the quintuple (E, A, B, C, D), and if $E \neq I$, then this condition may be satisfied if $\ker(D) \neq 0$, whereas it may *not* be satisfied if $\ker(D) = 0$; the problems in [14] - [15] appear to be regular [16]. More of this in the sequel.

A major role in our treatment will be reserved for the *Linear Matrix Inequality* (LMI) associated with an implicit system (1.9).

Definition 1.4.

Consider the matrix quintuple $\Sigma = (E, A, B, C, D)$, with $E, A \in \mathbb{R}^{l \times n}$, $B \in \mathbb{R}^{l \times m}$, $C \in \mathbb{R}^{r \times n}$, $D \in \mathbb{R}^{r \times m}$. Then a matrix $K = K' \in \mathbb{R}^{l \times l}$ satisfies the *Linear Matrix Inequality* (LMI) if $F(K) \geq 0$, with

$$F(K) := \begin{bmatrix} C'C + A'KE + E'KA & E'KB + C'D \\ B'KE + D'C & D'D \end{bmatrix}. \tag{1.10}$$

The set of solutions of the LMI is denoted by Γ:

$$\Gamma := \{K \in \mathbb{R}^{l \times l} \mid K = K',\ F(K) \geq 0\}. \tag{1.11}$$

First, we will define in Section 3 the concept of *Dissipation Inequality* (DI) for *general* systems (1.9), and optimal costs for LQCPs subject to (1.9) turn out to satisfy the DI (if these costs are finite). Then, the DI will be related to the LMI, and it will be shown that the concept of rank minimizing solutions of the LMI exists for *arbitrary* E, even if $sE - A$ is not invertible.

In Section 4, then, we will prove that optimal costs for LQCPs subject to implicit systems can be represented by means of certain solutions of the LMI, *regardless* whether the problems are *regular* or *not*. In addition, we will state conditions that are *necessary and sufficient* for finiteness of these optimal costs.

Thus, characterizations of optimal costs for LQCPs subject to *implicit* systems can be expressed *directly* in terms of the *original* system coefficients (E, A, B, C, D) for *arbitrary* E, as it is done for the case $E = I$ in Proposition 1.1. However, solutions of the LMI that correspond to these optimal costs need *not* be rank minimizing, as they are in the standard case (Proposition 1.1)! What is more, the algebraic Riccati equation [15], [17]

$$C'C + A'KE + E'KA - (E'KB + C'D)(D'D)^{-1}(B'KE + D'C) = 0 \tag{1.12}$$

may *not* have solutions even if $\ker(D) = 0$ and the LQCPs are solvable.

In fact, it turns out that characterizations of optimal costs for LQCPs subject to *general* linear systems (1.9) must be given in terms of *all* solutions of the LMI rather than in terms of rank minimizing ones - yet these characterizations *reduce* to the ones in Proposition 1.1 if $E = I$. In other words, Γ itself rather than the set of rank minimizing solutions of Γ turns out to be the *pivot* in linear-quadratic control subject to arbitrary linear systems. Several examples are included to underline all relevant aspects of our statements.

Finally, we discuss the *existence* of optimal inputs and optimal state trajectories for the problems under consideration. If these problems are well posed, then (possibly distributional) optimal controls and optimal state trajectories *always exist* for the LQCP *without* stability and (possibly distributional) optimal controls and state trajectories exist for the LQCP *with* stability if and only if Rosenbrock's system matrix [18] satisfies a certain rank condition, as is the case if $E = I$ [6]. Moreover, these inputs and state trajectories are *unique* if and only if the system matrix is left invertible. The actual *determination* of optimal controls and state trajectories can be quite involving, especially for nonsquare systems, and therefore we have shifted it to a subsequent paper.

2. Preliminaries.

The celebrated [10] contains the first distributional treatment of singular LQCPs subject to standard systems. The class of distributions $\mathcal{C}_{imp}$ in [10] is not only large enough to solve these problems [10], [6], [8], [19], but also adequate for describing *implicit* system properties such as geometric structure [20], feedback control and pole placement [21] - [22] and invertibility properties [23] - [24]. Since, in addition, $\mathcal{C}_{imp}$ has many nice properties, we see no reason for not adopting this class here; compare the allowed distributional class in [14].

The class $\mathcal{C}_{imp}$ is analyzed in detail in [25] and [10], see also [24] and, of course, Schwartz [9]. A distribution $u \in \mathcal{C}_{imp}$ is called *impulsive-smooth* and an impulsive-smooth distribution can be decomposed uniquely in an *impulse* (i.e., any linear combination of the Dirac delta distribution δ and its derivatives $\delta^{(i)}$, $i \geq 1$) and a *smooth* distribution. A distribution is called smooth if it corresponds to a function which is smooth on $\mathbb{R}^+$ and zero elsewhere. As in [10], a function f is smooth on $\mathbb{R}^+$ if $f(t)$ is arbitrarily often differentiable on $(0, \infty)$ and if, for all derivatives $f^{(i)}(t)$ $(i \geq 0)$, $\lim_{t \downarrow 0} f^{(i)}(t)$ exists and is finite.

The class $\mathcal{C}_{imp}$ is a commutative algebra over $\mathbb{R}$ with convolution $*$ of distributions as multiplication (unit element δ), and hence it is closed under differentiation (= convolution with $\delta^{(1)}$), and closed under integration (= convolution with H, the Heaviside distribution). It holds that $\delta^{(i)} = \delta^{(i-1)} * \delta^{(1)}$ $(i \geq 1)$, with $\delta^{(0)} = \delta$. By setting $\delta^{(-1)} := H$, $\delta^{(-j)} := \delta^{-(j-1)} * \delta^{(-1)}$ $(j \geq 1)$, we establish that $\delta^{(i+j)} = \delta^{(i)} * \delta^{(j)}$ $(i, j \in \mathbb{Z})$, and thus the inverse of $\delta^{(i)}$ (w.r.t. convolution), $(\delta^{(i)})^{-1}$, equals $\delta^{(-i)}$ $(i \in \mathbb{Z})$, $(\delta)^{-1} = \delta$, $\delta^{(-j)}$ is smooth, $\delta^{(-j)}(t) = t^{j-1}/(j-1)!$ $(t \in \mathbb{R}^+)$, $\delta^{(-j)}(t) = 0$ $(t < 0)$ for $j \geq 1$. If $\mathcal{C}_{p-imp}$, $\mathcal{C}_{sm} \in \mathcal{C}_{imp}$ denote the subalgebras of impulses and smooth distributions, respectively, then $\mathcal{C}_{imp} = \mathcal{C}_{p-imp} + \mathcal{C}_{sm}$. If $u \in \mathcal{C}_{sm}$, then $u(0^+) := \lim_{t \downarrow 0} u(t)$, and then the *distributional* derivative of u, $u^{(1)}$, equals $\dot{u} + u(0^+)\delta$, where $\dot{u}$ denotes the ordinary derivative of u on $\mathbb{R}^+$. Example: Let $u(t) = 2e^t$ on $\mathbb{R}^+$, $u(t) = 0$ on $(-\infty, 0)$. Then $u^{(1)} = u + 2\delta$, whereas $\dot{u} = u$. If $h \in \mathbb{R}$, then the distribution $\delta^{(1)} - h\delta$ is invertible; its smooth inverse equals e^{ht} on $\mathbb{R}^+$. In the sequel $\mathcal{C}_{imp}^k$ denotes the k-vector version of $\mathcal{C}_{imp}$ and $\mathcal{C}_{imp}^{k_1 \times k_2}$

the $k_1 \times k_2$ matrix with entries in $\mathcal{C}_{imp}$. If H is any real square matrix, then $(I\delta^{(1)} - H\delta)$ is invertible with inverse e^{Ht} on $\mathbb{R}^+$.

Let, finally, $\mathcal{C}_f$ denote the subalgebra of *fractional impulses*:

$$\mathcal{C}_f := \{u \in \mathcal{C}_{imp} | u = u_1 * u_2^{-1}, u_{1,2} \in \mathcal{C}_{p-imp}, u_2 \neq 0\}, \qquad (2.1)$$

then $\mathcal{C}_f$ is *isomorphic* to the field of rational functions $\mathbb{R}(s)$ [24, Proposition 2.3]. For instance, the polynomial $1 - s^2$ corresponds to the pulse $\delta - \delta^{(2)}$. The rational function $(s + 1)/(s - 3)$ corresponds to the fractional pulse $u = (\delta^{(1)} + \delta) * (\delta^{(1)} - 3\delta)^{-1}$. Set $v = (\delta^{(1)} - 3\delta)^{-1}$, then $u = \dot{v} + v(0^+)\delta + v$, with $v = e^{3t}$ on $\mathbb{R}^+$. Since $\dot{v} = 3v$, we get that $u = 4v + \delta$.

Due to the properties of $\mathcal{C}_{imp}$, we can keep the treatment fully *algebraic* by denoting convolution by juxtaposition and setting $p := \delta^{(1)}$, $p^0 := \delta$, $p^2 := \delta^{(2)}$, $p^{-1} := H$, and so forth (see [25] and [10]), and a multiple of δ, $\alpha\delta$, is then denoted by α, as $p^0 = 1$. The distributional derivative of $u \in \mathcal{C}_{imp}$, $u^{(1)} = \delta^{(1)} * u$, is replaced by pu, and thus $pu = \dot{u} + u(0^+)$, if $u \in \mathcal{C}_{sm}$.

Instead of (1.9), we now present its *distributional* version Σ:

$$pEx = Ax + Bu + Ex_0, \qquad (2.2a)$$

$$y = Cx + Du, \qquad (2.2b)$$

with $x_0 \in \mathbb{R}^n$ (and $Ex_0 = Ex_0\delta!$), $u \in \mathcal{C}^m_{imp}$. For every pair $(x_0, u) \in \mathbb{R}^n \times \mathcal{C}^m_{imp}$ we define the *solution set* [26, Section 2]

$$S(x_0, u) := \{x \in \mathcal{C}^n_{imp} | [pE - A]x = Bu + Ex_0\}. \qquad (2.3)$$

If x_0 is interpreted as $x(0^-)$, then (2.2a) coincides with the system equation in [14] as well as with (1.9a) in Laplace transform [27, § 22], [13], [20], [28]. Yet it should be stressed that (2.2a) is, in fact, an initial value problem for a linear differential-algebraic equation on $\mathbb{R}^+$ *in the distribution sense* [26].

If $sE - A$ is invertible, i.e., if $(pE - A)^{-1}$ exists, then for every pair $(x_0, u) \in \mathbb{R}^n \times \mathcal{C}^m_{imp}$ the solution set contains exactly one element: $x = (pE - A)^{-1}(Bu + Ex_0) \in \mathcal{C}^n_{imp}$. If, moreover, $E = I$ and $u \in \mathcal{C}^m_{sm}$, then this x coincides with the usual solution of (1.1a) on $\mathbb{R}^+$, as x(t) equals

$$e^{At}x_0 + {}_0\int^t e^{A(t - \tau)}Bu(\tau)d\tau$$

on $\mathbb{R}^+$ and $x(t) = 0$ if $t < 0$. Observe further that (2.2a) reduces to the distributional version of (1.1a) in [10], [6] if $E = I$.

Equation (2.2a) also reduces to (1.9a) if u as well as x are smooth; we then get $pEx = E\dot{x} + E(x(0^+)) = Ax + Bu + Ex_0$, and hence $E\dot{x} = Ax + Bu$ on $\mathbb{R}^+$, i.e., (1.9a). As a by-result, it follows that $x(0^+) - x_0 \in \ker(E)$ if $u \in C^m_{sm}$ and $x \in S(x_0, u) \cap C^n_{sm}$. This is a special case of

Lemma 2.1 [24, Main Lemma 2.5].

Let $x_0 \in \mathbb{R}^n$, $u \in C^m_{imp}$, $u = u_1 + u_2$, $u_1 \in C^m_{p-imp}$, $u_2 \in C^m_{sm}$, and $x \in S(x_0, u)$, $x = x_1 + x_2$, $x_1 \in C^n_{p-imp}$, $x_2 \in C^n_{sm}$. Then

$$pEx_1 + E(x(0^+)) = Ax_1 + Bu_1 + Ex_0, \tag{2.4a}$$

$$pEx_2 = Ax_2 + Bu_2 + E(x(0^+)). \tag{2.4b}$$

Example 1.3 (continued).

The system $p\begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} + \begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}\begin{bmatrix}x_{01}\\x_{02}\end{bmatrix}$ has as solutions $x_2 = 0$, $x_1 = -x_{02}$. If x_1 is smooth, then $x_{02} = 0$.

The LQCP *without* and *with* stability subject to (2.2) are then defined as follows.

Definition 2.2.

Consider Σ (2.2).

$(\mathrm{LQCP})^-$: For all $x_0 \in \mathbb{R}^n$, determine

$$J^-(x_0) := \inf\{{}_0\!\int^\infty y'(t)y(t)dt \,|\, u \in C^m_{sm},\ x \in S(x_0, u) \cap C^n_{sm}\}, \tag{2.5}$$

and if, for all x_0, $J^-(x_0) < \infty$, then for every x_0 compute (if possible) $u \in C^m_{sm}$ and $x \in S(x_0, u) \cap C^n_{sm}$ such that ${}_0\!\int^\infty y'(t)y(t)dt = J^-(x_0)$.

$(\mathrm{LQCP})^+$: For all $x_0 \in \mathbb{R}^n$, determine

$$J^+(x_0) := \inf\{{}_0\!\int^\infty y'(t)y(t)dt \,|\, u \in C^m_{sm},\ x \in S(x_0, u) \cap C^n_{sm},\ \lim_{t\to\infty} x(t) = 0\}, \tag{2.6}$$

and if, for all x_0, $J^+(x_0) < \infty$, then for every x_0 compute (if possible) $u \in C^m_{sm}$ and $x \in S(x_0, u) \cap C^n_{sm}$, $\lim_{t\to\infty} x(t) = 0$, such that ${}_0\!\int^\infty y'(t)y(t)dt = J^+(x_0)$.

These problems are called *regular* if, for all $x_0 \in \mathbb{R}^n$,

$$y \in C^r_{sm} \Leftrightarrow u \in C^m_{sm},\ x \in S(x_0, u) \cap C^n_{sm}, \tag{2.7}$$

and *singular* if (2.7) is not satisfied.

This definition of regularity for LQCPs subject to general systems (2.2) appears in [16, Definition 3.1], and it is clear that (2.7) is a system property. If $E = I$, then (2.7) is satisfied if and only if $\ker(D) = 0$ [16, Proposition 2.1]. For general E we have [16, Theorem 3.2].

Proposition 2.3.

The LQCPs in Definition 2.2 are regular if and only if

$$\ker(\begin{bmatrix} E & 0 \\ C & D \end{bmatrix}) \cap [A\ B]^{-1}\mathrm{im}(E) = 0. \tag{2.8}$$

Example 1.3 (continued and extended).

The LQCPs subject to $p\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}u + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_{01} \\ x_{02} \end{bmatrix}$, $y = u$ are *singular*, whereas $\ker(D) = 0$! Indeed, $u = 0$ is optimal for $(\mathrm{LQCP})^-$ as well as for $(\mathrm{LQCP})^+$, since $u = 0$ yields $x_2 = 0$, $x_1 = -x_{02}$, impulsive.

In the sequel we will frequently need several subspaces from [24].

Definition 2.4 [24, Definition 3.1], [26, Definition 4.1].

Consider the system Σ (2.2). A point $x_0 \in \mathbb{R}^n$ is called *weakly unobservable* if there exists an input $u \in C^m_{sm}$ and a state trajectory $x \in S(x_0, u) \cap C^n_{sm}$ such that $y = 0$. If, moreover, $x(0^+) = x_0$, then x_0 is called *weakly unobservable in the sense of consistency.* The space of former points is denoted by $\mathcal{V} = \mathcal{V}(\Sigma)$, the space of latter points by $\mathcal{V}_C = \mathcal{V}_C(\Sigma)$. A point $x_0 \in \mathbb{R}^n$ is called *strongly controllable* if there exists an input $u \in C^m_{p\text{-}imp}$ and a state trajectory $x \in C^n_{p\text{-}imp}$ such that $y = 0$. The space of these points is denoted by $\mathcal{W} = \mathcal{W}(\Sigma)$. A point $x_0 \in \mathbb{R}^n$ is called *consistent* if there exists an input $u \in C^m_{sm}$ and a state trajectory $x \in S(x_0, u) \cap C^n_{sm}$ such that $x(0^+) = x_0$. The space of these points is denoted by $I_C = I_C(\Sigma)$. Finally, a point $x_0 \in \mathbb{R}^n$ is called *weakly consistent* if there exists an input $u \in C^m_{sm}$ such that $S(x_0, u) \cap C^n_{sm} \neq \emptyset$. The space of these points is denoted by $I^w_C = I^w_C(\Sigma)$.

Proposition 2.5 [24, Proposition 3.4, Theorem 3.6, Theorem 3.9].

$\mathcal{V}_C$ is the largest subspace $\mathcal{L}$ for which $\begin{bmatrix} A \\ C \end{bmatrix}\mathcal{L} \subset \begin{bmatrix} E\mathcal{L} \\ 0 \end{bmatrix} + \mathrm{im}(\begin{bmatrix} B \\ D \end{bmatrix})$, and $\mathcal{V} = \mathcal{V}_C + \ker(E)$. $\mathcal{W}$ is the smallest subspace $\mathcal{K}$ for which

$$E^{-1}[A\ B]\{(\mathcal{K} \oplus \mathbb{R}^m) \cap \ker([C\ D])\} \subset \mathcal{K}.$$

I_C is the largest subspace $\mathcal{M}$ for which $A\mathcal{M} \subset E\mathcal{M} + \mathrm{im}(B)$, and $I_C^W = I_C + \ker(E)$.

Algorithms for computing all spaces are in [24, Proposition 3.8, Theorem 3.10], and it follows that $\mathcal{V}_C = \begin{bmatrix} A \\ C \end{bmatrix}^{-1}\{\begin{bmatrix} E\mathcal{V}_C \\ 0 \end{bmatrix} + \mathrm{im}(\begin{bmatrix} B \\ D \end{bmatrix})\}$, $I_C = A^{-1}\{EI_C + \mathrm{im}(B)\}$, $\mathcal{W} = E^{-1}[A\ B]\{(\mathcal{W} \oplus \mathbb{R}^m) \cap \ker([C\ D])\}$.

From Definitions 2.2 and 2.4 we observe that $(LQCP)^-$ and $(LQCP)^+$ are *not* solvable if $I_C^W \neq \mathbb{R}^n$; if $x_0 \notin I_C^W$, then we will define $J^+(x_0) := +\infty$, $J^-(x_0) := +\infty$. Hence a *necessary* condition for solvability of any LQCP in Definition 2.2 is: $I_C^W = \mathbb{R}^n$.

Example 2.6.

Consider the system

$$p\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}u + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_{01} \\ x_{02} \end{bmatrix}, \quad y = [0\ 1]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

It follows that $x_1 = 0$, $x_2 + u + x_{01} = 0$, and hence $I_C^W = \ker(E)$; if $x_{01} \neq 0$, then for every x_{02} and every smooth u, $S(\begin{bmatrix} x_{01} \\ x_{02} \end{bmatrix}, u) \cap C^2_{sm} = \emptyset$. For such points x_0 we have $J^+(x_0) = +\infty$, $J^-(x_0) = +\infty$. However, if

$$J_d^+(x_0) := \inf\{{}_0\!\int^{\infty} y'(t)y(t)dt \mid u \in C^m_{imp},\ x \in S(x_0, u),\ \lim_{t\to\infty} x(t) = 0\},$$

then $J_d^+(x_0) = 0$, as $u = -x_{01}$ (impulsive) yields $x_1 = 0$, $x_2 = 0$. We establish that infimization over functions *may exceed* infimization over distributions if $E \neq I$; it is proven in [19, Proposition 2.24] that this is *not* the case if $E = I$. More of this in Section 4.

3. The Dissipation Inequality and the Linear Matrix Inequality.

Given the distributional system Σ:

$$pEx = Ax + Bu + Ex_0, \tag{3.1a}$$

$$y = Cx + Du, \tag{3.1b}$$

with $x_0 \in \mathbb{R}^n$, $u \in \mathcal{C}^m_{imp}$, and the solution set $S(x_0, u)$ (2.3).

In this Section we will define the Dissipation Inequality (DI) for an implicit system (3.1), and it will be shown that the optimal costs (2.5) - (2.6) satisfy the DI. Then, we will unravel the link between DI and LMI, and investigate Γ (1.11), the set of solutions of the LMI.

Let the subspaces $\mathfrak{X}_1$, $\mathfrak{X}_3$, $\mathfrak{X}_4$ be such that $\mathfrak{X}_1 \oplus (I_C \cap \ker(E)) = I_C$, $(I_C \cap \ker(E)) \oplus \mathfrak{X}_3 = \ker(E)$, $I^W_C \oplus \mathfrak{X}_4 = \mathbb{R}^n$, and let the subspace $\mathcal{L}_2$ be such that $EI_C \oplus \mathcal{L}_2 = \mathbb{R}^l$. In addition, let $\mathcal{U}_2$ be such that $\mathcal{U}_1 \oplus \mathcal{U}_2 = \mathbb{R}^m$, with $\mathcal{U}_1 = B^{-1}(EI_C)$. Then, w.r.t. suitably chosen bases, (3.1a) decomposes into

$$p\begin{bmatrix} E_{11} & 0 & 0 & E_{14} \\ 0 & 0 & 0 & E_{24} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ A_{21} & A_{22} & A_{23} & A_{24} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} B_{11} & B_{12} \\ 0 & B_{22} \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} +$$

$$\begin{bmatrix} E_{11} & 0 & 0 & E_{14} \\ 0 & 0 & 0 & E_{24} \end{bmatrix}\begin{bmatrix} x_{01} \\ x_{02} \\ x_{03} \\ x_{04} \end{bmatrix}, \tag{3.2a}$$

with $\ker(B_{22}) = 0$, and E_{11} invertible. Since there exists a $F \in \mathbb{R}^{m\times n}$ such that $(A + BF)I_C \subset EI_C$ [24, Proposition 3.5, Theorem 3.6], it follows that $A_{21} = B_{22}F_{21}$, $A_{22} = B_{22}F_{22}$ for some matrices F_{21}, F_{22}. Moreover, by construction and Lemma 1 in the Appendix,

$$[-A_{23},\ sE_{24} - A_{24},\ -B_{22}] \text{ is left unimodular.} \tag{3.2b}$$

Finally, note that $[E_{24}\ A_{23}\ A_{24}\ B_{22}]$ is right invertible if $[E\ A\ B]$ is assumed to be of full row rank.

Without loss of generality, we may (and hence will) assume $[E\ A\ B]$ to be of full row rank. The next result generalizes [21, Theorem 2].

Proposition 3.1 [26, Theorem 4.5].

Let $[E\ A\ B]$ be of full row rank. Then $I^W_C = \mathbb{R}^n$ if and only if

$$im(E) + im(B) + A[\ker(E)] = \mathbb{R}^l. \tag{3.3}$$

The condition (3.3) can be interpreted as controllability in the sense of Verghese [13], or as *impulse controllability* [35]; the latter will become evident in Proposition 3.5.

In the remainder of this paper (3.3) will be a *standing* assumption, as it is obviously *necessary* for solvability of our LQCPs in Definition 2.2. This has nice implications for (3.2a); it reduces to

$$p\begin{bmatrix} E_{11} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} B_{11} & B_{12} \\ 0 & B_{22} \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} E_{11} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_{01} \\ x_{02} \\ x_{03} \end{bmatrix} \tag{3.4}$$

with E_{11} invertible, $A_{21} = B_{22}F_{21}$, $A_{22} = B_{22}F_{22}$, and $[A_{23}\ \ B_{22}]$ *invertible*, due to (3.2b).

We will demonstrate that *all* possible system trajectories for (3.1) can now be expressed in system trajectories for a *standard* system of reduced order $e = \text{rank}\ (E)$, and *vice versa*.

For, if x_{01}, x_{02}, x_{03}, u_1, u_2, x_1, x_2 and x_3 satisfy (3.4), then

$$x_3 = 0,\ u_2 = -F_{21}x_1 - F_{22}x_2, \tag{3.5}$$

and thus, with $\bar{A}_{11} := (A_{11} - B_{12}F_{21})$, $\bar{A}_{12} := (A_{12} - B_{12}F_{22})$, (3.6)

$$px_1 = E_{11}^{-1}\bar{A}_{11}x_1 + E_{11}^{-1}[\bar{A}_{12}\ \ B_{11}]\begin{bmatrix} x_2 \\ u_1 \end{bmatrix} + x_{01}, \tag{3.7}$$

a standard system equation (1.1a) of size $e = \text{rank}\ (E)$.

If (3.1b) is partitioned in accordance with (3.4):

$$y = [C_1\ C_2\ C_3]\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + [D_1\ D_2]\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \tag{3.8}$$

then (3.8) becomes

$$y = \bar{C}_1x_1 + [\bar{C}_2\ \ D_1]\begin{bmatrix} x_2 \\ u_1 \end{bmatrix}, \tag{3.9}$$

with $\bar{C}_1 = C_1 - D_2F_{21}$, $\bar{C}_2 = C_2 - D_2F_{22}$. (3.10)

Now, let $\bar{x}(\bar{x}_0, \begin{bmatrix} \bar{v}_1 \\ \bar{v}_2 \end{bmatrix})$ denote the unique solution of

$$p\bar{x} = E_{11}^{-1}\bar{A}_{11}\bar{x} + E_{11}^{-1}[\bar{A}_{12}\ \ B_{11}]\begin{bmatrix} \bar{v}_1 \\ \bar{v}_2 \end{bmatrix} + \bar{x}_0, \tag{3.11}$$

with $\begin{bmatrix} \bar{v}_1 \\ \bar{v}_2 \end{bmatrix} \in C_{imp}^{n+m-1}$ and $\bar{x}_0 \in \mathbb{R}^e$, and

$$\bar{y}(\bar{x}_0, \begin{bmatrix} \bar{v}_1 \\ \bar{v}_2 \end{bmatrix}) := \bar{C}_1\bar{x}(\bar{x}_0, \begin{bmatrix} \bar{v}_1 \\ \bar{v}_2 \end{bmatrix}) + [\bar{C}_2\ \ D_1]\begin{bmatrix} \bar{v}_1 \\ \bar{v}_2 \end{bmatrix}. \tag{3.12}$$

Proposition 3.2.

Let $[x_{01}', x_{02}', x_{03}']' \in \mathbb{R}^n$ be given, let $[u_1', u_2']' \in C_{imp}^m$, and let $[x_1', x_2', x_3']' \in C_{imp}^n$ satisfy (3.4) and (3.8). Then $x_3 = 0$, $u_2 = -F_{21}x_1 - F_{22}x_2$. In addition, $x_1 = \bar{x}(x_{01}, \begin{bmatrix} x_2 \\ u_1 \end{bmatrix})$ and $y = \bar{y}(x_{01}, \begin{bmatrix} x_2 \\ u_1 \end{bmatrix})$.

Conversely, if $\bar{x}_0 \in \mathbb{R}^e$, $\begin{bmatrix}\bar{v}_1\\ \bar{v}_2\end{bmatrix} \in C^{n+m-1}_{imp}$, then the controls $u_1 = \bar{v}_2$ and $u_2 = - F_{21}\bar{x}(\bar{x}_0, \begin{bmatrix}\bar{v}_1\\ \bar{v}_2\end{bmatrix}) - F_{22}\bar{v}_1$ and the state trajectories $x_1 = \bar{x}(\bar{x}_0, \begin{bmatrix}\bar{v}_1\\ \bar{v}_2\end{bmatrix})$, $x_2 = \bar{v}_1$ and $x_3 = 0$ satisfy (3.4) for $x_{01} = \bar{x}_0$ and x_{02}, x_{03} arbitrary. In addition, for these choices of controls and state trajectories the output y in (3.8) equals $\bar{y}(\bar{x}_0, \begin{bmatrix}\bar{v}_1\\ \bar{v}_2\end{bmatrix})$.

Proof. If in (3.11) - (3.12) we insert $\begin{bmatrix}\bar{v}_1\\ \bar{v}_2\end{bmatrix} = \begin{bmatrix}x_2\\ u_1\end{bmatrix}$ with $\bar{x}_0 = x_{01}$, then $p(\bar{x} - x_1) = E_{11}^{-1}\bar{A}_{11}(\bar{x} - x_1)$ ((3.7)) and hence $\bar{x} = x_1$, $y = \bar{y}(\bar{x}_0, \begin{bmatrix}\bar{v}_1\\ \bar{v}_2\end{bmatrix})$. The converse is immediate.

Proposition 3.2 is a more geometrically orientated version of [29, Theorem 2.1], which generalizes a state space decomposition in [15], where sE - A is assumed to be invertible. Proposition 3.2 will be of great use to us in the sequel.

In our definition of the *Dissipation Inequality* we will need a few extra concepts.

Definition 3.3.

Let T > 0. Then a function $f: [0, T] \to \mathbb{R}^k$ is called *smooth on [0, T]* if $f \in C^{\infty}((0, T) \to \mathbb{R}^k)$ and if, for all $i \geq 0$, $\lim_{t \downarrow 0} f^{(i)}(t)$ and $\lim_{t \uparrow T} f^{(i)}(t)$ exist and are finite.

Definition 3.4.

Consider the implicit system

$$Ex'(t) = Ax(t) + Bu(t) \tag{3.13}$$

on $[0, \infty)$, with $E, A \in \mathbb{R}^{l\times n}$, $B \in \mathbb{R}^{l\times m}$. Let T > 0. For every $x_0 \in \mathbb{R}^n$ and every smooth u on [0, T] we define the *set of smooth solutions* on [0, T]

$$S^T_{sm}(x_0, u) := \{x: [0, T] \to \mathbb{R}^n \mid x \text{ smooth on } [0, T], \forall_{t \in (0, T)}: (3.13) \text{ holds}, x(0^+) - x_0 \in \ker(E)\}. \tag{3.14}$$

Then a point $x_0 \in \mathbb{R}^n$ will be called *weakly consistent on [0, T]* if there exists a smooth input u on [0, T] such that $S^T_{sm}(x_0, u) \neq \emptyset$. If, in addition, for some $x \in S^T_{sm}(x_0, u)$, $x(0^+) = x_0$, then x_0 will be called *consistent on [0, T]*. The system (3.13) is *control-solvable on [0, T] in the function sense* if *every* $x_0 \in \mathbb{R}^n$ is weakly consistent on [0, T].

Definition 3.4 is the analogon for a finite time interval of [26, Definitions 3.1, 4.1]; the definition of control solvability is rooted in the observation that in many control problems x_0, interpreted as $x(0^-)$, may be arbitrary, as a result of which one might be interested in designing control laws that work for *all* possible states values rather than for a certain subset. If $E = I$, then every $x_0 \in \mathbb{R}^n$ is obviously consistent on $[0, T]$. In general, (3.13) is control-solvable on $[0, T]$ in the function sense for *every* $T > 0$ if and only if (3.3) is satisfied, i.e., if (3.1a) is impulse controllable [35]. For a short proof of this, see Proposition 3.5.

Proposition 3.5.

Let $[E\ A\ B]$ be of full row rank. Then

$\forall_{T > 0}$: (3.13) is control-solvable in the function sense on $[0, T]$

$\Leftrightarrow \mathrm{im}(E) + \mathrm{im}(B) + A[\ker(E)] = \mathbb{R}^l$.

Proof. $\Leftarrow$ According to Proposition 3.1, $I_C^w = \mathbb{R}^n$. Then, for every $x_0 \in \mathbb{R}^n$ there exists an input $u \in C_{sm}^m$ and a state trajectory $x \in C_{sm}^n$ such that $pEx = Ax + Bu + Ex_0$. It follows that $Ex'(t) = Ax(t) + Bu(t)$ on $[0, \infty)$ with $x(0^+) - x_0 \in \ker(E)$ and thus x_0 is control-solvable in the function sense for every $T > 0$. $\Rightarrow$ Assume that $\eta B = 0$, $\eta E = 0$ and $\eta A\bar{x} = 0$ for all $\bar{x} \in \ker(E)$. Let $T > 0$ and $x_0 \in \mathbb{R}^n$, arbitrary. Then there exists a smooth function x: $[0, T] \to \mathbb{R}^n$ such that $\eta Ax(0^+) = 0$ and thus $\eta Ax_0 = \eta Ax(0^+) + \eta A[x_0 - x(0^+)] = 0$; hence $\eta A = 0$ and we establish that $\eta = 0$.

Definition 3.6.

Consider the implicit system (1.9). Then a function V: $\mathbb{R}^n \to \mathbb{R}$ satisfies the *Dissipation Inequality* (DI) for (1.9) if

$V(x_0) = 0$ for all $x_0 \in \ker(E)$,

and if for all $x_0 \in \mathbb{R}^n$, for all $T \geq 0$, for all smooth u on $[0, T]$ and for all $x \in S_{sm}^T(x_0, u)$,

$$\int_0^T y'(t)y(t)dt + V(x(T^-)) \geq V(x_0) \tag{3.15}$$

(with $x(0^-) := x_0$).

Theorem 3.7.

If, for all $x_0 \in \mathbb{R}^n$, $J^-(x_0) < \infty$, then $J^-: \mathbb{R}^n \to \mathbb{R}^+$ satisfies the DI for (1.9). Moreover, for every $x_0 \in \mathbb{R}^n$ and every $T \geq 0$,

$$J^-(x_0) = \inf\{{}_0\int^T y'(t)y(t)dt + J^-(x(T^-)) \mid u \text{ smooth on } [0, T], \; x \in S^T_{sm}(x_0, u)\} \qquad (3.16)$$

(with $x(0^-) := x_0$), and there exists a unique positive semidefinite $P^- \in \mathbb{R}^{n\times n}$, with $\ker(E) \subset \ker(P^-)$, such that, for all $x_0 \in \mathbb{R}^n$, $J^-(x_0) = x_0'P^-x_0$. If, for all $x_0 \in \mathbb{R}^n$, $J^+(x_0) < \infty$, then $J^+: \mathbb{R}^n \to \mathbb{R}^+$ satisfies the DI for (1.9). Moreover, J^+ satisfies (3.16) for every $x_0 \in \mathbb{R}^n$ and every $T > 0$, and there exists a unique positive semidefinite $P^+ \in \mathbb{R}^{n\times n}$, with $\ker(E) \subset \ker(P^+)$, such that, for all $x_0 \in \mathbb{R}^n$, $J^+(x_0) = x_0'P^+x_0$.

Proof. We only consider the case $T > 0$. Due to Proposition 3.5, (3.13) is control-solvable in the function sense on [0, T] for *every* $T > 0$. Let the system (1.9) be decomposed as Σ is in (3.4) and (3.8), and let $x_0 \in \mathbb{R}^n$, $T > 0$, u be smooth on [0, T] and $x \in S^T_{sm}(x_0, u)$. Then it follows that $x_{1,2,3}$ and $u_{1,2}$ in (3.4) are smooth and $x_1(0^+) = x_{01}$. In fact, on [0, T] we have (3.5), (3.7) and (3.9). Conversely, if $\bar{x}_0 \in \mathbb{R}^e$, $\begin{bmatrix}\bar{v}_1\\ \bar{v}_2\end{bmatrix}$ is smooth on [0, T], and $\bar{x}$ denotes the resulting solution on [0, T] of (3.11), then $\begin{bmatrix}u_1\\ u_2\end{bmatrix}$, with $u_1 = \bar{v}_2$, $u_2 = -F_{21}\bar{x} - F_{22}\bar{v}_1$, is smooth on [0, T], $x_1 = \bar{x}$, $x_2 = \bar{v}_1$ and $x_3 = 0$ are smooth, $x_1(0^+) = \bar{x}_0$, and y(t) equals $\bar{y}(\bar{x}_0, \begin{bmatrix}\bar{v}_1\\ \bar{v}_2\end{bmatrix})(t)$ on [0, T].
By Proposition 3.2, $J^-(x_0) = \bar{J}^-(x_{01})$, $J^-(x(T^-)) = \bar{J}^-(x_1(T^-))$, where $\bar{J}^-(\bar{x}_0) := \inf\{{}_0\int^\infty \bar{y}'(t)\bar{y}(t)dt \mid \begin{bmatrix}\bar{v}_1\\ \bar{v}_2\end{bmatrix} \in C^{n+m-1}_{sm}\}$, the optimal cost for $(LQCP)^-$ subject to the standard system (3.11) - (3.12). Since $\bar{J}^-(x_{01}) < \infty$ for every x_{01}, the function $\bar{J}^-: \mathbb{R}^e \to \mathbb{R}^+$ satisfies the associated DI (1.5), and also

$$\bar{J}^-(x_{01}) = \inf\{{}_0\int^T \bar{y}'(t)\bar{y}(t)dt + \bar{J}^-(\bar{x}(T^-)) \mid \begin{bmatrix}\bar{v}_1\\ \bar{v}_2\end{bmatrix} \text{ smooth on } [0, T]\}$$

[3, Lemma 1]. Hence, again by Proposition 3.2 and the above, $J^-: \mathbb{R}^n \to \mathbb{R}^+$ satisfies the DI as well as (3.16). Moreover, since there exists a unique positive semidefinite $\bar{P}^-$ such that, for all x_{01}, $\bar{J}^-(x_{01}) = x_{01}'\bar{P}^-x_{01}$ if $\bar{J}^-(x_{01}) < \infty$ for every $x_{01} \in \mathbb{R}^e$ [7], [30], we establish that there exists a unique positive semidefinite $P^- \in \mathbb{R}^{n\times n}$, with $\ker(E) \subset \ker(P^-)$, such that, for all $x_0 \in \mathbb{R}^n$, $J^-(x_0) = x_0'P^-x_0$.

Next, let $\bar{J}^+(x_{01}) := \inf\{{}_0\int^{\infty}\bar{y}'(t)\bar{y}(t)dt \mid \begin{bmatrix}\bar{v}_1\\ \bar{v}_2\end{bmatrix} \in C_{sm}^{n+m-1},\ \lim_{t\to\infty}\bar{x}(t) = 0\}$, and assume that $J^+(x_0) < \infty$ for every x_0. Then $\bar{J}^+(x_{01}) < \infty$ for every x_{01} (Proposition 3.2), and there exists a unique $\bar{P}^+ \geq 0$ such that, for all x_{01}, $\bar{J}^+(x_{01}) = x_{01}'\bar{P}^+x_{01}$ (Proposition 1.1) and, consequently, there exists a unique $P^+ \geq 0$, with $\ker(E) \subset \ker(P^+)$, such that, for all x_0, $J^+(x_0) = x_0'P^+x_0$. The remaining claims for J^+ are clear by the foregoing.

The DI can be interpreted for implicit *dissipative* systems, as it was done in [2] - [3] for (1.5) with respect to standard dissipative systems; we will not elaborate on such issues here. Observe that (3.16) reflects the *Bellman* optimality principle [31], see also [3].

Next, we will link the DI (3.15) with F(K) (1.10). Lemma 3.8 generalizes a classical result for standard systems [32], [1].

Lemma 3.8.

Consider (1.9) and let $K = K' \in \mathbb{R}^{l\times l}$, $x_0 \in \mathbb{R}^n$, $T \geq 0$, u smooth on $[0, T]$ and $x \in S_{sm}^T(x_0, u)$. If $x(0^-) := x_0$, then

$$
{}_0\int^T y'(t)y(t)dt + x'(T^-)E'KEx(T^-) = x_0'E'KEx_0 + {}_0\int^T [x'(t)\ u'(t)][F(K)]\begin{bmatrix}x(t)\\ u(t)\end{bmatrix}dt. \tag{3.17}
$$

Proof. We have $y'(t)y(t) + (d/dt)(x'(t)EKEx(t)) = y'(t)y(t) + [x'(t)A' + u'(t)B']KEx(t) + x'(t)E'K[Ax(t) + Bu(t)] = [x'(t)\ u'(t)][F(K)]\begin{bmatrix}x(t)\\ u(t)\end{bmatrix}$ on $(0, T)$ if $T > 0$. The rest is straightforward.

Now, assume that the matrix P^- (Theorem 3.7) exists. Then there exists a symmetric matrix $K^- \in \mathbb{R}^{l\times l}$ such that $P^- = E'K^-E$, as $\ker(E) \subset \ker(P^-)$. Such a matrix K^- may *not* be unique; it may *not* be positive semidefinite either. If in terms of (3.4), $P^- = \begin{bmatrix}P_{11}^- & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$, with $P_{11}^- \geq 0$, then *every* $K^- = \begin{bmatrix}K_{11}^- & K_{12}\\ K_{12}' & K_{22}\end{bmatrix}$, with $K_{11}^- = (E_{11}')^{-1}P_{11}^-E_{11}^{-1}$ and K_{22} symmetric, satisfies the requirements. Analogously, if P^+ in Theorem 3.7 exists, then there exists a symmetric $K^+ \in \mathbb{R}^{l\times l}$ such that $P^+ = E'K^+E$, and K^+ may be nonunique and indefinite. Theorem 3.9 translates Theorem 3.7 in K^- and K^+ by means of Lemma 3.8.

Theorem 3.9.

Assume that $J^-(x_0) < \infty$ for all $x_0 \in \mathbb{R}^n$, and let $P^- \geq 0$ be such that $J^-(x_0) = x_0'P^-x_0$ for all x_0. Then for all $K^- = (K^-)' \in \mathbb{R}^{l\times l}$ that satisfy $P^- = E'K^-E$, for all $x_0 \in \mathbb{R}^n$, for all $T \geq 0$, for all smooth u on [0, T] and for all $x \in S^T_{sm}(x_0, u)$, it holds that

$$ {}_0\!\int^T [x'(t)\ u'(t)][F(K^-)]\begin{bmatrix} x(t) \\ u(t) \end{bmatrix} dt \geq 0. \tag{3.18}$$

Moreover, for all such K^-, all $x_0 \in \mathbb{R}^n$ and all $T \geq 0$,

$$\inf\{ {}_0\!\int^T [x'(t)\ u'(t)][F(K^-)]\begin{bmatrix} x(t) \\ u(t) \end{bmatrix} dt \mid u \text{ smooth on } [0, T],\ x \in S^T_{sm}(x_0, u)\} = 0. \tag{3.19}$$

Assume that $J^+(x_0) < \infty$ for all $x_0 \in \mathbb{R}^n$, and let $P^+ \geq 0$ be such that $J^+(x_0) = x_0'P^+x_0$ for all x_0. Then analogous statements hold for all $K^+ = (K^+)' \in \mathbb{R}^{l\times l}$ that satisfy $P^+ = E'K^+E$.

Let us investigate (3.18) further by using the decomposition in (3.4). Let $T > 0$, $x_0 \in \mathbb{R}^n$, u be smooth on [0, T] and $x \in S^T_{sm}(x_0, u)$. Then $x_{1,2,3}$ and $u_{1,2}$ are all smooth, $x_1(0^+) = x_{01}$, and $x_3 = 0$, $u_2 = -F_{21}x_1 - F_{22}x_2$. Let $K^- = \begin{bmatrix} K_{11}^- & K_{12}^- \\ (K_{12}^-)' & K_{22}^- \end{bmatrix}$. Then (3.18) reduces to

$$ {}_0\!\int^T [x_1'(t),\ x_2'(t),\ u_1'(t)][\bar{F}_1(K_{11}^-)]\begin{bmatrix} x_1(t) \\ x_2(t) \\ u_1(t) \end{bmatrix} dt \geq 0, \tag{3.20}$$

with $\bar{F}_1(K_{11}) :=$

$$\begin{bmatrix} \bar{C}_1'\bar{C}_1 + \bar{A}_{11}'K_{11}E_{11} + E_{11}'K_{11}\bar{A}_{11} & E_{11}'K_{11}[\bar{A}_{12}\ B_{11}] + \bar{C}_1'[\bar{C}_2\ D_1] \\ \begin{bmatrix} \bar{A}_{12}' \\ B_{11}' \end{bmatrix} K_{11}E_{11} + \begin{bmatrix} \bar{C}_2' \\ D_1' \end{bmatrix}\bar{C}_1 & \begin{bmatrix} \bar{C}_2' \\ D_1' \end{bmatrix}[\bar{C}_2\ D_1] \end{bmatrix} \tag{3.21}$$

(see (3.6) and (3.10)). It follows from Theorem 3.9 that (3.20) is valid for *all* x_{01}, *all* $T \geq 0$, and *all* smooth $\begin{bmatrix} x_2 \\ u_1 \end{bmatrix}$ on [0, T], with $x_1(t)$ the smooth solution of (3.11) on [0, T] (see the proof of Theorem 3.9). Consequently [3, Lemma 4], [33, p. 799],

$$\bar{F}_1(K_{11}^-) \geq 0, \tag{3.22}$$

and (3.22) is *necessary* for the claim:

$$F(K^-) \geq 0 \tag{3.23}$$

for some symmetric K^- that satisfies $P^- = E'K^-E$. This follows directly from the observation that $F(K^-) \geq 0$ if and only if $\bar{F}(K^-) \geq 0$, with $\bar{F}(K)$

$$= \bar{F}(\begin{bmatrix} K_{11} & K_{12} \\ K_{12}' & K_{22} \end{bmatrix}) = \begin{bmatrix} & \bar{F}_1(K_{11}) & \\ (\begin{bmatrix} A_{13}' \\ B_{12}' \end{bmatrix}K_{11} + \begin{bmatrix} A_{23}' \\ B_{22}' \end{bmatrix}K_{12}')E_{11} + \begin{bmatrix} C_3' \\ D_2' \end{bmatrix}\bar{C}_1 & \begin{bmatrix} C_3' \\ D_2' \end{bmatrix}[\bar{C}_2\ D_1] \end{bmatrix}$$

$$\left.\begin{array}{ll} & E_{11}'(K_{11}[A_{13}\ B_{12}] + K_{12}[A_{23}\ B_{22}]) + \bar{C}_1'[C_3\ D_2] \\ & \qquad\qquad\qquad\qquad\qquad\qquad \begin{bmatrix} \bar{C}_2' \\ D_1' \end{bmatrix}[C_3\ D_2] \\ \begin{bmatrix} C_3' \\ D_2' \end{bmatrix}[C_3\ D_2] & \end{array}\right]. \quad (3.24)$$

Lemma 3.10.

Let $P_{[\bar{C}_2\ D_1]} := I - [\bar{C}_2\ D_1]\left(\begin{bmatrix} \bar{C}_2' \\ D_1' \end{bmatrix}[\bar{C}_2\ D_1]\right)^+ \begin{bmatrix} \bar{C}_2' \\ D_1' \end{bmatrix}$, with M^+ denoting the Moore-Penrose inverse of any matrix M. If $\bar{F}_1(K_{11}^-) \geq 0$, and K_{12}^- is chosen such that $E_{11}'(K_{11}^-[A_{13}\ B_{12}] + K_{12}^-[A_{23}\ B_{22}]) =$

$$E_{11}'K_{11}^-[\bar{A}_{12}\ B_{11}]\left(\begin{bmatrix} \bar{C}_2' \\ D_1' \end{bmatrix}[\bar{C}_2\ D_1]\right)^+ \begin{bmatrix} \bar{C}_2' \\ D_1' \end{bmatrix}[C_3\ D_2] - \bar{C}_1'P_{[\bar{C}_2\ D_1]}[C_3\ D_2], \quad (3.25)$$

then $\bar{F}(K^-) \geq 0$.

Proof. Appendix; recall that $[A_{23}\ B_{22}]$ is invertible and hence there exists for every K_{11}^- exactly one K_{12}^- such that (3.25) is satisfied.

Lemma 3.10 shows that P^- can be expressed in an element of Γ, the set of solutions of the LMI (Definition 1.4). Similarly, P^+ in Theorem 3.9 can be related to an element of Γ. In other words, we have found the relation between DI and LMI.

Theorem 3.11.

Assume that, for all $x_0 \in \mathbb{R}^n$, $J^-(x_0) < \infty$ and let $P^- \geq 0$ be such that, for all x_0, $J^-(x_0) = x_0'P^-x_0$. Then there exists a $K^- \in \Gamma$ such that $P^- = E'K^-E$. If. for all $x_0 \in \mathbb{R}^n$, $J^+(x_0) < \infty$, and $P^+ \geq 0$ is such that, for all x_0, $J^+(x_0) = x_0'P^+x_0$, then there exists a $K^+ \in \Gamma$ such that $P^+ = E'K^+E$.

Proof. If $P^- = E'K^-E$, with K^- symmetric, and $K^- = \begin{bmatrix} K_{11}^- & K_{12}^- \\ (K_{12}^-)' & K_{22}^- \end{bmatrix}$, then $\bar{F}_1(K_{11}^-) \geq 0$ (3.22), and if K_{12}^- is chosen in accordance with (3.25), then $\bar{F}(K^-) \geq 0$ (Lemma 3.10) and hence $F(K^-) \geq 0$. The proof for P^+ runs similar.

If $E = I$, then $K^- = P^-$ and $K^+ = P^+$ in Theorem 3.11, and we recover the relation between the DI and the LMI for standard systems [1], [3].

In the remainder of this Section we will derive two interesting results for Γ (1.11), the set of solutions of the LMI. First, we must borrow from [24] the concept of right-invertibility in the strong sense.

Definition 3.12 [24, Definition 4.7].

The system Σ (3.1) is *right invertible in the strong sense* if

$$\forall_{x_0 \in \mathbb{R}^n} \forall_{\bar{y} \in C^r_{imp}} \exists_{u \in C^m_{imp}} \exists_{x \in S(x_0,\, u)} : y = \bar{y}.$$

The recent [24] contains two definitions for right- as well as for left-invertibility of a general system (3.1); in [24, Definition 4.3] the system Σ is called right invertible in the *weak* sense if

$$\forall_{\bar{y} \in C^r_{imp}} \exists_{u \in C^m_{imp}} \exists_{x \in S(0,\, u)} : y = \bar{y},$$

and the two concepts coincide if $\det(sE - A) \neq 0$ [23] - [24].

Proposition 3.13 [24, Corollary 4.13].

Assume that $[E \;\; A \;\; B]$ is of full row rank. Then the following statements are equivalent.

i) Σ is right invertible in the strong sense.

ii) $\mathcal{V}(\Sigma) + \mathcal{W}(\Sigma) = \mathbb{R}^n$, $\begin{bmatrix} E & A & B \\ 0 & C & D \end{bmatrix}$ is of full row rank.

iii) $\begin{bmatrix} A - sE & B \\ C & D \end{bmatrix}$ is right invertible as a rational matrix.

Observe that right-invertibility (in either sense) is equivalent to right-invertibility of the transfer function

$$T(s) := D + C(sE - A)^{-1}B \tag{3.26}$$

[23] if $\det(sE - A) \neq 0$. If $E = I$, we reobtain [10, Theorem 3.24].

Next, let $\kappa :=$ normal rank $\begin{bmatrix} A - sE & B \\ C & D \end{bmatrix}$, (3.27)

i.e., κ denotes the normal rank of Rosenbrock's system matrix.

Theorem 3.14.

Assume that (3.3) holds. Then, for every $K \in \Gamma$,

$$\text{rank }(F(K)) \geq \kappa - 1. \tag{3.28}$$

Let Σ_K denote the system

$$pEx = Ax + Bu + Ex_0,\; y_K = C_K x + D_K u, \tag{3.29}$$

with $F(K) = \begin{bmatrix} C_K' \\ D_K' \end{bmatrix}[C_K \;\; D_K]$, rank $(F(K))$ = rank $([C_K \;\; D_K])$. Then rank $(F(K))$ $= \kappa - 1$ if and only if Σ_K is right invertible in the strong sense.

Proof. Appendix.

Let $\Gamma_{\min}$ denote the set of *rank minimizing* elements of Γ:

$$\Gamma_{\min} := \{K \in \Gamma \,|\, \text{rank } (F(K)) = \kappa - l\}. \tag{3.30}$$

If $\det (sE - A) \neq 0$, then $\kappa = l + \rho$, with $\rho :=$ normal rank $(T(s))$ (3.26), and $\Gamma = \{K \in \Gamma \,|\, \text{rank } (F(K)) = \rho\}$. If $E = I$, then we reobtain the key result in [4]. Recall that $\rho = m$ if $E = I$ and $\ker(D) = 0$. In general we have

Corollary 3.15.

Assume that (2.8) holds. Then $\kappa = n + m$, and $\Gamma_{\min} = \{K \in \Gamma \,|\, \text{rank } (F(K)) = n + m - l\}$. If (2.8) holds, $\ker(D) = 0$, $l = n$, and $\phi(K) :=$

$$C'C + A'KE + E'KA - (E'KB + C'D)(D'D)^{-1}(B'KE + D'C), \tag{3.31}$$

then $\Gamma_{\min} = \{K = K' \in \mathbb{R}^{l\times l} \,|\, \phi(K) = 0\}$.

Proof. By means of the decomposition in (3.4) and (3.8) it is easily seen that (2.8) is valid if and only if $[\bar{C}_2 \; D_1]$, with $\bar{C}_2 = C_2 - D_2F_{22}$, is left invertible. It follows that Rosenbrock's system matrix is left invertible as a rational matrix, and thus $\kappa = n + m$. If, in addition, $l = n$ (E and A are square), and $\ker(D) = 0$, then $\Gamma_{\min} = \{K \in \Gamma \,|\, \text{rank } (F(K)) = m\}$ on one hand, and on the other, by Schur's lemma [12], $\text{rank } (F(K)) = \text{rank } (\phi(K)) + \text{rank } (D'D) = \text{rank } (\phi(K)) + m$. Thus, $\text{rank } (F(K)) = m$ if and only if $\phi(K) = 0$.

Observe that Corollary 3.15 reduces to the well-known statement that the rank minimizing solutions of the LMI are the symmetric solutions of the algebraic Riccati equation (1.8) if $E = I$ and $\ker(D) = 0$ (see Section 1). We saw in Proposition 1.1 that optimal costs for LQCPs subject to *standard* systems are represented by rank minimizing solutions of the associated LMI, and hence the set of solutions of (1.8) is of interest for these LQCPs if $\ker(D) = 0$.

However, in Section 4 we will see that for *general* E the algebraic Riccati equation (ARE) $\phi(K) = 0$ (3.31) need *not* have solutions *even if* (2.8) holds, $l = n$ and $\ker(D) = 0$. In particular, this implies that optimal costs for LQCPs subject to *implicit* systems need *not* be represented by rank minimizing solutions of the LMI. On the other hand, the ARE $\phi(K) = 0$ may have solutions if these LQCPs are *singular*, i.e., if (2.8) does not hold (Proposition 2.3).

Example 1.3 (continued and extended).

If $K = \begin{bmatrix} k_{11} & k_{12} \\ k_{12} & k_{22} \end{bmatrix}$, then $F(K) \geq 0$ if and only if $k_{11} = 0$ and $k_{12} \in [0, 2]$. Observe that (2.8) does not hold. Yet $\ker(D) = 0$, and the ARE is defined; its solutions satisfy $k_{11} = 0$, and $k_{12} = 0$ or 2. Note that $\kappa = n + m = 3$, although (2.8) is violated, and that $\kappa - 1 = \kappa - n = m = 1$.

In Section 4 we will prove that "suitable" K^- and K^+ in Theorem 3.11 must be selected from *all* elements of Γ (1.11) rather than from all elements of $\Gamma_{\min}$ (3.30). Yet our characterizations for K^- and K^+ will reduce to those in Proposition 1.1 if $E = I$! An essential tool in our selection procedure is Lemma 3.16, an obvious generalization of a crucial statement in [8, Theorem 6.2]. As in [8], the strongly controllable subspace $\mathcal{W} = \mathcal{W}(\Sigma)$ (Definition 2.4) turns out to play a major role in describing solutions of the LMI.

Lemma 3.16.

Assume that (3.3) holds. Then

$$K \in \Gamma \Rightarrow \mathcal{W}(\Sigma) \subset \ker(E'KE).$$

Proof. The strongly controllable subspace $\mathcal{W} = \mathcal{W}(\Sigma)$ can be computed by the algorithm $\mathcal{W}_0 = \ker(E)$, $\mathcal{W}_{i+1} = E^{-1}[A\ B]\{(\mathcal{W}_i \oplus \mathbb{R}^m) \cap \ker([C\ D])\}$; $\mathcal{W}_n = \mathcal{W}$ [24, Theorem 3.10]. Obviously, $\mathcal{W}_0 \subset \ker(E'KE)$. Now, let $x_0 \in \mathcal{W}_1$. Then there exist $\bar{x}$ and $\bar{u}$ such that $Ex_0 = A\bar{x} + B\bar{u}$, $C\bar{x} + D\bar{u} = 0$, $E\bar{x} = 0$. Hence $[\bar{x}'\ \ \bar{u}'][F(K)]\begin{bmatrix}\bar{x}\\ \bar{u}\end{bmatrix} = 0$, and thus $[F(K)]\begin{bmatrix}\bar{x}\\ \bar{u}\end{bmatrix} = \begin{bmatrix}E'KEx_0\\ 0\end{bmatrix} = 0$ and $x_0 \in \ker(E'KE)$. Assume that $\mathcal{W}_i \subset \ker(E'KE)$ $(i \geq 0)$, and let $x_0 \in \mathcal{W}_{i+1}$. Then there exist $\bar{x} \in \mathcal{W}_i$ and $\bar{u}$ such that $Ex_0 = A\bar{x} + B\bar{u}$, $C\bar{x} + D\bar{u} = 0$. Again, $[\bar{x}'\ \ \bar{u}'][F(K)]\begin{bmatrix}\bar{x}\\ \bar{u}\end{bmatrix} = 0$, and hence $[F(K)]\begin{bmatrix}\bar{x}\\ \bar{u}\end{bmatrix} = \begin{bmatrix}E'KEx_0 + A'KE\bar{x}\\ B'KE\bar{x}\end{bmatrix} = 0$. Since $\operatorname{im}(E) + \operatorname{im}(B) + A[\ker(E)] = \mathbb{R}^l$, it follows that $E'KEx_0 = 0$ and this completes the proof by induction.

4. Existence and characterization of optimal costs; existence and uniqueness of optimal inputs.

This Section deals with *necessary and sufficient* conditions for existence of $J^-(x_0)$ and $J^+(x_0)$ (2.5) - (2.6), with *characterization of P^- and P^+* (Theorem 3.7) in terms of the LMI (Definition 1.4), and with *existence and uniqueness* of optimal inputs.

As in [29], the system Σ (3.1) will be called *output stabilizable* if

$$\forall_{x_0} \in \mathbb{R}^n \ \exists_u \in C^m_{sm} \ \exists_x \in S(x_0, u) \cap C^n_{sm} : \lim_{t\to\infty} y(t) = 0. \tag{4.1}$$

In addition, we will call Σ *state stabilizable* if

$$\forall_{x_0} \in \mathbb{R}^n \ \exists_u \in C^m_{sm} \ \exists_x \in S(x_0, u) \cap C^n_{sm} : \lim_{t\to\infty} x(t) = 0. \tag{4.2}$$

Proposition 4.1 [29, Theorem 2.2].

Let [E A B] be of full row rank and let $\mathcal{V} = \mathcal{V}(\Sigma)$ denote the weakly unobservable subspace. Then Σ is output stabilizable if and only if (3.3) is satisfied and if, for all $\lambda \in \bar{\mathbb{C}}^+ := \{s \in \mathbb{C} | \mathrm{Re}(s) \geq 0\}$,

$$(\eta[\lambda E - A, - B] = 0, \ \eta E\mathcal{V} = 0) \Leftrightarrow \eta = 0. \tag{4.3}$$

Moreover, Σ is output stabilizable if and only if, for every $x_0 \in \mathbb{R}^n$, $J^-(x_0) < \infty$.

For standard systems, condition (4.3) appears in [30] as a generalization of the usual Hautus criterion for *state* stabilizability. For *arbitrary* E we have

Corollary 4.2.

Let [E A B] be of full row rank. Then Σ is state stabilizable if and only if (3.3) holds and

$$\forall_s \in \bar{\mathbb{C}}^+ : [sE - A, - B] \text{ is right invertible.} \tag{4.4}$$

Moreover, Σ is state stabilizable if and only if, for every $x_0 \in \mathbb{R}^n$, $J^+(x_0) < \infty$.

Proof. If in Proposition 4.1, C = I and D = 0, then $\mathcal{V}_C = \mathcal{V}_C(\Sigma) = 0$ and hence $E\mathcal{V} = E\mathcal{V}_C = 0$ [Definition 2.4, Proposition 2.5], and (4.3) reduces to (4.4). Finally, it is clear that $J^+(x_0) < \infty$ for every $x_0 \in \mathbb{R}^n$ only if Σ is state stabilizable. Conversely, if Σ is state stabilizable, then it follows directly from (3.4) and (3.8) that (3.11) - (3.12) is state stabilizable, and hence, for every $\bar{x}_0 \in \mathbb{R}^e$, the optimal cost for $(LQCP)^+$ subject to (3.11) - (3.12) is finite (e.g. [6]). By Proposition 3.2, then, $J^+(x_0) < \infty$ for every $x_0 \in \mathbb{R}^n$, and the proof is complete.

Now, let us, besides (2.5) - (2.6), introduce for the system Σ the functions

$$J_d^-(x_0) := \inf\{{}_0\int^\infty y'(t)y(t)dt \mid u \in C_{imp}^m,\ x \in S(x_0, u)\}, \tag{4.5}$$

and $$J_d^+(x_0) := \inf\{{}_0\int^\infty y'(t)y(t)dt \mid u \in C_{imp}^m,\ x \in S(x_0, u),\ \lim_{t\to\infty} x(t) = 0\}. \tag{4.6}$$

Here, ${}_0\int^\infty y'(t)y(t)dt$ is set equal to $+\infty$ if either $y \notin C_{sm}^r$ or $y \in C_{sm}^r$ and y is not square-integrable over $\mathbb{R}^+$. Also, $\lim_{t\to\infty} x(t)$ stands for $\lim_{t\to\infty} x_2(t)$ if x_2 denotes the smooth component of $x \in C_{imp}^n$. Then, obviously, $J_d^-(x_0) \leq J^-(x_0)$, $J_d^+(x_0) \leq J^+(x_0)$, and strict inequality may occur - in Example 2.6 we saw that $J_d^+(x_0) = 0$ and $J^+(x_0) = +\infty$ for certain points x_0. Note that this Example does *not* satisfy (3.3), our standing assumption since Section 3. In this final Section we will establish that infimization over functions *equals* infimization over distributions if (3.3) is satisfied, and optimal inputs and state trajectories for (4.5) - (4.6) are in general distributions, unless (2.8) holds.

Lemma 4.3.

Assume that $x_0 \in \mathbb{R}^n$, $u \in C_{sm}^m$ and $x \in S(x_0, u) \cap C_{sm}^n$. Then

$${}_0\int^\infty y'(t)y(t)dt < \infty \Rightarrow \lim_{t\to\infty} d(x(t), \mathcal{V} + \mathcal{W}) = 0,$$

with $d(\bar{x}, \mathcal{L})$ denoting the (Euclidean) distance between $\bar{x} \in \mathbb{R}^n$ and the subspace $\mathcal{L} \subset \mathbb{R}^n$.

Proof. Consider (3.4) and (3.8). Then, by (3.5), we get (3.7) and (3.9) and hence the Euclidean distance between $x_1(t)$ and $\bar{\mathcal{V}} + \bar{\mathcal{W}}$ converges to zero, as time tends to infinity [19, Corollary 3.28], [30, Remark 1]. Here, $\bar{\mathcal{V}}$ and $\bar{\mathcal{W}}$ denote the weakly unobservable and strongly controllable subspace for (3.11) - (3.12), respectively. The claim then follows from Proposition 3.2.

Theorem 4.4.

Assume that [E A B] is of full row rank, and consider $(LQCP)^-$. For all $x_0 \in \mathbb{R}^n$, $J^-(x_0) < \infty$ if and only if Σ is output stabilizable. Assume this to be the case. Then there exists a unique $P^- \in \mathbb{R}^{n\times n}$, $P^- \geq 0$, such that, for all $x_0 \in \mathbb{R}^n$, $J_d^-(x_0) = J^-(x_0) = x_0'P^-x_0$. For some $K^- \in \Gamma$, $P^- = E'K^-E$, and $E'KE \leq E'K^-E$ if $K \in \Gamma$ and $\mathcal{V} \subset E'KE$. Furthermore, $\ker(P^-) = \mathcal{V} + \mathcal{W}$. For every $x_0 \in \mathbb{R}^n$ there exists an input $u \in C^m_{imp}$ and a state trajectory $x \in S(x_0, u)$ such that $y \in C^r_{sm}$ and $x_0'P^-x_0 = {}_0\int^\infty y'(t)y(t)dt$. If (2.8) holds, then these optimal inputs and optimal state trajectories are functions of the *Bohl* type, i.e., linear combinations of functions of the type $t^k e^{\lambda t}$, $k \geq 0$.

Proof. The first claim follows from Proposition 4.1. Assume that Σ is output stabilizable. Then Theorem 3.7 guarantees the existence of P^- such that, for all x_0, $J^-(x_0) = x_0'P^-x_0$. By Theorem 3.11, there exists a $K^- \in \Gamma$ such that $P^- = E'K^-E$. Now suppose that, for some $\bar{x}_0$, $J_d^-(\bar{x}_0) < \bar{x}_0'P^-\bar{x}_0$. Then there exists an input $u \in C^m_{imp}$ and a state trajectory $x \in S(x_0, u)$ such that $y \in C^r_{sm}$ and ${}_0\int^\infty y'(t)y(t)dt < \bar{x}_0'P^-\bar{x}_0$. If $u = u_1 + u_2$, $u_1 \in C^m_{p-imp}$, $u_2 \in C^m_{sm}$ and $x = x_1 + x_2$, $x_1 \in C^n_{p-imp}$, $x_2 \in C^n_{sm}$, then $Cx_1 + Du_1 = 0$ and $pEx_1 = Ax_1 + Bu_1 + E[\bar{x}_0 - x(0^+)]$ (2.4a), and hence, by definition, $x_0 - x(0^+) \in \mathcal{W} = \mathcal{W}(\Sigma)$. Consequently, by Lemma 3.16, ${}_0\int^\infty y'(t)y(t)dt < x(0^+)P^-x(0^+)$, as $y = Cx_2 + Du_2$ and $pEx_2 = Ax_2 + Bu_2 + E[x(0^+)]$ (2.4b), and we have a contradiction with (2.5). Thus, $J_d^-(x_0) = J^-(x_0)$ for every x_0. Now the *existence* of optimal inputs and state trajectories within the class of impulsive-smooth distributions is stated in [29, Theorem 2.3]; also, it is shown in [29] that these inputs and state trajectories are Bohl functions if (2.8) holds.

Next, it follows from Lemma 3.16 that $\mathscr{V} + \mathscr{W} \subset \ker(P^-)$, since, obviously, $\mathscr{V} \subset \ker(P^-)$. On the other hand, if $x_0'P^-x_0 = 0$, then there *exists* an input $u \in C^m_{imp}$ and a trajectory $x \in S(x_0, u)$ such that $y = 0$, and hence, by [24, Theorem 3.2], $x_0 \in \mathscr{V} + \mathscr{W}$. Thus, $\ker(P^-) = \mathscr{V} + \mathscr{W}$. Finally, let $K \in \Gamma$ and $\mathscr{V} \subset \ker(E'KE)$. By Lemma 3.16, then, $\mathscr{V} + \mathscr{W} \subset \ker(E'KE)$. Let $x_0 \in \mathbb{R}^n$ be arbitrary, $u \in C^m_{sm}$ and $x \in S(x_0, u) \cap C^n_{sm}$ such that ${}_0\int^{\infty} y'(t)y(t)dt < \infty$. Then, necessarily,

$$\lim_{t\to\infty} x'(t)E'KEx(t) = 0 \tag{4.7}$$

(Lemma 4.3), and hence, by (3.17),

$${}_0\int^{\infty} y'(t)y(t)dt = x_0'E'KEx_0 + {}_0\int^{\infty}[x'(t)\ \ u'(t)][F(K)]\begin{bmatrix}x(t)\\u(t)\end{bmatrix}dt \tag{4.8}$$

with $F(K) \geq 0$. Thus, ${}_0\int^{\infty} y'(t)y(t)dt \geq x_0'E'KEx_0$ and $J^-(x_0) = x_0'P^-x_0 = x_0'E'K^-Ex_0 \geq x_0'E'KEx_0$ for every x_0, as a result of which $E'K^-E \geq E'KE$. This completes the proof.

The characterization for P^- in Theorem 4.4 determines P^- *uniquely;* if for some other matrix $\tilde{P} \geq 0$ there exists a $\tilde{K} \in \Gamma$ such that $\tilde{P} = E'\tilde{K}E$, $\mathscr{V} \subset \ker(\tilde{P})$, and $E'KE \leq \tilde{P}$ for any $K \in \Gamma$ that satisfies $\mathscr{V} \subset \ker(E'KE)$, then $P^- = E'K^-E \leq \tilde{P}$ and $\tilde{P} = E'\tilde{K}E \leq P^-$.

Corollary 4.5.

Assume that $E = I$ and that Σ is output stabilizable. Then $J^-(x_0) = x_0'K^-x_0$, with $K^- \in \Gamma$, $\mathscr{V} + \mathscr{W} = \ker(K^-)$, and if $K \in \Gamma$ with $K\mathscr{V} = 0$, then $K \leq K^-$.

Remark 4.6.

In [19, Proposition 3.20] it is demonstrated that the characterizations for K^- in Proposition 1.1 and in Corollary 4.5 coincide. Unlike Proposition 1.1, Corollary 4.5 represents K^- in terms of *all* solutions of the LMI, rather than in rank minimizing ones.

Theorem 4.7.

Assume that $[E\ A\ B]$ is of full row rank, and consider $(LQCP)^+$. Then $J^+(x_0) < \infty$ for every $x_0 \in \mathbb{R}^n$ if and only if Σ is state stabilizable. Assume this to be the case. Then there exists a unique $P^+ \in \mathbb{R}^{n\times n}$, $P^+ \geq 0$, such that, for all $x_0 \in \mathbb{R}^n$, $J_d^+(x_0) = J^+(x_0) = x_0'P^+x_0$. For some $K^+ \in \Gamma$, $P^+ = E'K^+E$, and $E'KE \leq E'K^+E$ for all $K \in \Gamma$. For every $x_0 \in \mathbb{R}^n$, there exists an input $u \in C_{imp}^m$ and a state trajectory $x \in S(x_0, u)$ such that $y \in C_{sm}^r$ and $x_0'P^+x_0 = {}_0\int^\infty y'(t)y(t)dt$ if and only if

$$\forall_\lambda \in \{s \in \mathbb{C} | \mathrm{Re}(s) = 0\}: \mathrm{rank} \begin{bmatrix} A - sE & B \\ C & D \end{bmatrix} = \kappa. \tag{4.9}$$

These optimal inputs and optimal state trajectories are Bohl functions if (2.8) holds.

Proof. The first statement follows from Corollary 4.2. Then, by Theorems 3.7 and 3.11, there exists a unique positive semidefinite P^+ such that, for all $x_0 \in \mathbb{R}^n$, $J^+(x_0) = x_0'P^+x_0$, and $P^+ = E'K^+E$ for some $K^+ \in \Gamma$. Hence $J_d^+(x_0) = x_0'P^+x_0$ (see proof of Theorem 4.4). Now, let $K \in \Gamma$, $x_0 \in \mathbb{R}^n$, $u \in C_{sm}^m$, $x \in S(x_0, u) \cap C_{sm}^n$, $\lim_{t\to\infty} x(t) = 0$ and ${}_0\int^\infty y'(t)y(t)dt < \infty$. Then, by (3.17), we get (4.8) and hence $x_0'P^+x_0 = x_0'E'K^+Ex_0 \geq x_0'E'KEx_0$; i.e., $E'K^+E \geq E'KE$.

Next, it is well known that for every initial condition an optimal impulsive-smooth input (with impulsive-smooth state trajectory) exists for $(LQCP)^+$ subject to a *standard* system if and only if there are no *invariant zeros* on the imaginary axis (e.g. [6]). These invariant zeros can be characterized as those $s \in \mathbb{C}$ for which the rank of Rosenbrock's system matrix is smaller than κ, its normal rank (3.27) [34]. Now, consider (3.4), (3.8). It follows directly that, for every $s \in \mathbb{C}$,

$$\mathrm{rank} \begin{bmatrix} A - sE & B \\ C & D \end{bmatrix} = \mathrm{rank} \begin{bmatrix} E_{11}{}^{-1}\bar{A}_{11} - sI & E_{11}{}^{-1}[\bar{A}_{12}\ B_{11}] \\ \bar{C}_1 & [\bar{C}_2\ \ D_1] \end{bmatrix} + 1 - e,$$

as $[A_{23}\ B_{22}]$ is invertible, and hence, by Proposition 3.2, for every $x_0 \in \mathbb{R}^n$ there exists an input $u \in C_{imp}^m$ and a state trajectory $x \in S(x_0, u)$, with $\lim_{t\to\infty} x(t) = 0$, such that $y \in C_{sm}^r$ and $x_0'P^+x_0 = {}_0\int^\infty y'(t)y(t)dt$ if and only if (4.9) is satisfied. Finally, (2.8) holds if and only if $[\bar{C}_2\ D_1]$ in (3.11) - (3.12) is left invertible, and optimal inputs and state trajectories for a regular $(LQCP)^+$ subject to a standard system are Bohl functions (e.g. [6]). Then, the proof is completed by, again, applying Proposition 3.2.

Also the representation of $P^+ = E'K^+E$ in Theorem 4.7 is *unique*; if, for some other $\tilde{P} \geq 0$, $J^+(x_0) = x_0'\tilde{P}x_0$ for all x_0, with $\tilde{P} = E'\tilde{K}E$, $\tilde{K} \in \Gamma$, and $E'\tilde{K}E \geq E'KE$ for all $K \in \Gamma$, then $\tilde{P} = E'\tilde{K}E \leq P^+$ and $P^+ = E'K^+E \leq \tilde{P}$. If $E = I$, then we reobtain the representation of K^+ in Proposition 1.1.

Our Theorems 4.4 and 4.7 thus present *necessary and sufficient* conditions for existence of (2.5) and (2.6); moreover, if these conditions are satisfied, then (2.5) and (2.6) can be represented by certain solutions of the LMI.

Example 4.8 [14] - [15].

Consider the system

$$p\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} + \begin{bmatrix}0\\1\end{bmatrix}u + \begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}\begin{bmatrix}x_{01}\\x_{02}\end{bmatrix},$$

$$y = \begin{bmatrix}1 & 0\\0 & 1\\0 & 0\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} + \begin{bmatrix}0\\0\\1\end{bmatrix}u.$$

Condition (2.7) is obviously satisfied, and hence we have (2.8) (Proposition 2.3). Also, the system is already in the decomposition (3.4), (3.8), and $\mathcal{X}_3$ and $\mathcal{U}_1$ are not appearing. Note that $\det(sE - A) \neq 0$. For (3.11) - (3.12) we get $px_1 = x_2 + x_{01}$, $y = \begin{bmatrix}1\\0\\-1\end{bmatrix}x_1 + \begin{bmatrix}0\\1\\0\end{bmatrix}x_2$. The ARE for this subsystem is $2 - k^2 = 0$ with $k = 2^{1/2}$ as only positive semidefinite solution. Hence $(LQCP)^-$ as well as $(LQCP)^+$ for this subsystem have the same optimal cost as well as the same optimal inputs, since the associated system matrix is clearly left invertible for every $s \in \mathbb{C}$. These optimal inputs are $x_2 = -2^{1/2}x_1$ with resulting trajectory $x_1(t) = e^{-2^{1/2}t}x_{01}$ on $\mathbb{R}^+$. Thus, for the *original* system the optimal input and optimal state trajectory follow from the system equation itself and from

$$2^{1/2}x_1 + x_2 = 0,$$

and the optimal cost for $(LQCP)^-$ as well as for $(LQCP)^+$ is $\begin{bmatrix}k_{11} & 0\\0 & 0\end{bmatrix}$, with $k_{11} = 2^{1/2}$. Let us check this by using the Theorems 4.4 and 4.7. If $K = \begin{bmatrix}k_{11} & k_{12}\\k_{12} & k_{22}\end{bmatrix}$, then $F(K) = \begin{bmatrix}1 + 2k_{12} & k_{11} & k_{12}\\k_{11} & 1 & 0\\k_{12} & 0 & 1\end{bmatrix}$, and thus

$$\Gamma = \{K \mid 2 - (k_{12} - 1)^2 - k_{11}^2 \geq 0\};$$

it follows that $k_{11} \le 2^{1/2}$, and $K^+ = \begin{bmatrix} 2^{1/2} & 1 \\ 1 & k_{22} \end{bmatrix}$ (k_{22} arbitrary) $\in \Gamma$. Hence, for every $K \in \Gamma$, $E'KE = \begin{bmatrix} k_{11} & 0 \\ 0 & 0 \end{bmatrix} \le E'K^+E = \begin{bmatrix} 2^{1/2} & 0 \\ 0 & 0 \end{bmatrix} = P^+$, by Theorem 4.7. In addition, by Theorem 4.4, $P^- = P^+$, since $\mathcal{V} = \ker(E)$.

Next, it is readily checked that κ (3.27) equals 3 and hence $\kappa - 1 = 1$ (observe that $\kappa - 1 = \rho$, with ρ = normal rank $(T(s))$ (3.26), since $\det(sE - A) \neq 0$); yet, for *every* $K \in \Gamma$ we have rank $(F(K)) \geq 2$! Thus, $\Gamma_{\min} = \emptyset$ (3.30). Equivalently, by Corollary 3.15, the ARE $\phi(K) = 0$ (3.31) has *no* solutions, even although the LQCPs under consideration are solvable [15]. We establish that $\Gamma_{\min}$ is of importance w.r.t. LQCPs if E is invertible (Proposition 1.1), but $\Gamma_{\min}$ may be *empty* if E is *singular*. Hence "suitable" matrices for characterization of (2.5) and (2.6) are to be searched among elements of Γ instead of $\Gamma_{\min}$, *regardless* whether E is invertible or not, as is shown by the Theorems 4.4 and 4.7.

Finally, let us take a look at the system Σ_{K^+} (3.29), with $F(K^+)$ factorized as $\begin{bmatrix} C_{K^+}' \\ D_{K^+}' \end{bmatrix} [C_{K^+} \; D_{K^+}]$. We have $F(K^+) = \begin{bmatrix} 3 & 2^{1/2} & 1 \\ 2^{1/2} & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$, and $C_{K^+} = \begin{bmatrix} 2^{1/2} & 1 \\ 1 & 0 \end{bmatrix}$, $D_{K^+} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ satisfy the requirements. We establish immediately that $\mathcal{V}(\Sigma_{K^+}) = \mathbb{R}^2$; for every $\begin{bmatrix} x_{01} \\ x_{02} \end{bmatrix}$, the input $u = -x_1$ is such the trajectory following from $x_2 = -2^{1/2}x_1$ and $px_1 = x_2 + x_{01}$ is smooth and $y_{K^+} = 0$. As K^+ is *not* rank minimizing, it follows from Theorem 3.14 that Σ_{K^+} is *not* right invertible in the strong sense, as a result of which $\begin{bmatrix} E & A & B \\ 0 & C_{K^+} & D_{K^+} \end{bmatrix}$ is *not* right invertible (Proposition 3.13). This is indeed the case.

Example 4.9.

Consider the system

$$p\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & -1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_{01} \\ x_{02} \\ x_{03} \end{bmatrix},$$

$$y = [0 \; 0 \; 0]\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + [1 \; 0]\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}.$$

We have $e = 2$, $l = 4$, $n = 3$, $\mathrm{im}(E) + \mathrm{im}(B) = \mathbb{R}^4$, $x_0 \in \mathcal{V}$ if and only if $x_{02} = 0$, $x_0 \in \mathcal{W}$ if and only if $x_{01} = 0$. Also, (2.8) does *not* hold. In (3.4), (3.8), $\mathscr{x}_3$ and $\mathscr{u}_1$ are not appearing. The subsystem (3.11), (3.12) becomes

$$p\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} = \begin{bmatrix}1 & 1\\ 0 & -1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}0\\ 1\end{bmatrix}x_3 + \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}x_{01}\\ x_{02}\end{bmatrix},$$
$$y = [0\ 1]\begin{bmatrix}x_1\\ x_2\end{bmatrix}.$$

The LQCPs associated with this subsystem are (indeed) *singular*. It follows from Proposition 1.1 that $K^- = 0$ and that $K^+ = \begin{bmatrix}2 & 0\\ 0 & 0\end{bmatrix}$. Optimal controls for $(\mathrm{LQCP})^-$ as well as for $(\mathrm{LQCP})^+$ turn out to be impulsive. For the former problem $x_3 = -x_{02}$ is optimal; this impulse yields $x_2 = 0$ and thus $y = 0$. For the latter problem $x_3 = -x_{02} - 2x_{01}$ is optimal, since the resulting x_2 equals $-2(p+1)^{-1}x_{01}$ and thus $x_1 = (p-1)^{-1}[-2(p+1)^{-1} + 1]x_{01} = (p-1)^{-1}[(p-1)(p+1)^{-1}]x_{01} = (p+1)^{-1}x_{01}$, $x_{1,2}$ converge to zero as time tends to infinity, and ${}_0\!\int^{\infty} y'y\,dt = 2x_{01}^2$.

Now, let $K = \begin{bmatrix}k_{11} & k_{12} & k_{13} & k_{14}\\ k_{12} & k_{22} & k_{23} & k_{24}\\ k_{13} & k_{23} & k_{33} & k_{34}\\ k_{14} & k_{24} & k_{34} & k_{44}\end{bmatrix}$. Then $F(K)$ (1.10) equals

$$\begin{bmatrix}2k_{11} & k_{13} & k_{13} - 2k_{14} & k_{11} - k_{13} & k_{12} - k_{13} + 2k_{14}\\ k_{13} & -2k_{22} + 2k_{23} & k_{23} - 2k_{24} & k_{12} - k_{23} & k_{22} - k_{23} + 2k_{24}\\ k_{13} - 2k_{14} & k_{23} - 2k_{24} & 0 & 0 & 0\\ k_{11} - k_{13} & k_{12} - k_{23} & 0 & 1 & 0\\ k_{12} - k_{13} + 2k_{14} & k_{22} - k_{23} + 2k_{24} & 0 & 0 & 0\end{bmatrix}$$

and $F(K) \geq 0$ if and only if

$$\begin{bmatrix}2k_{11} & k_{11} & k_{12} & k_{11} - k_{13} & k_{12} - k_{13} + 2k_{14}\\ k_{11} & 2k_{12} - 2k_{22} + 1 & k_{22} & k_{12} - k_{23} + 1 & k_{22} - k_{23} + 2k_{24}\\ k_{12} & k_{22} & 0 & 0 & 0\\ k_{11} - k_{13} & k_{12} - k_{23} + 1 & 0 & 1 & 0\\ k_{12} - k_{13} + 2k_{14} & k_{22} - k_{23} + 2k_{24} & 0 & 0 & 0\end{bmatrix}$$

≥ 0, and this is the case only if $k_{12} = 0$, $k_{22} = 0$, $k_{13} = 2k_{14}$, $k_{23} = 2k_{24}$. Moreover, $k_{11}(2 - k_{11}) \geq 0$ and hence $k_{11} \in [0, 2]$. It is easily seen that $K^+ \in \Gamma$ if $k_{11}{}^+ = 2$, $k_{13}{}^+ = 2$ and $k_{23}{}^+ = 1$, and also that $K^- \in \Gamma$ with $k_{11}{}^- = 0$, $k_{13}{}^- = 0$, $k_{23}{}^- = 0$. Thus, by Theorems 4.4 and 4.7, $P^- = 0$ and $P^+ = \begin{bmatrix}2 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$, in accordance with the above.

Finally, $F(K^-) = \begin{bmatrix}0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 0\end{bmatrix}$, $F(K^+) = \begin{bmatrix}4 & 2 & 0 & 0 & 0\\ 2 & 2 & 0 & -1 & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & -1 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 0\end{bmatrix}$, and appropriate factorizations for (3.29) are given by $[C_{K^-}\ D_{K^-}] = [[0\ 0\ 0]\ [1\ 0]]$ and

$[C_{K^+}\ D_{K^+}] = \left[\begin{bmatrix}2 & 1 & 0\\ 0 & 1 & 0\end{bmatrix}\ \begin{bmatrix}0 & 0\\ -1 & 0\end{bmatrix}\right]$. Note that rank $(F(K^-)) = 1 = \kappa - 1$ (3.27), whereas rank $(F(K^+)) = 2$. Hence $K^- \in \Gamma_{min}$ (3.30), and, indeed, $\mathcal{V}(\Sigma_{K^-}) + \mathcal{W}(\Sigma_{K^-}) = \mathbb{R}^3$ ($u_1 = x_2$, $u_2 = -x_{02}$ yield $x_2 = 0$, $u_1 = 0$ and hence $y_{K^-} = 0$) and $\begin{bmatrix}E & A & B\\ 0 & C_{K^-} & D_{K^-}\end{bmatrix}$ is of full row rank, in accordance with Proposition 3.13 and Theorem 3.14. Moreover, $\mathcal{V}(\Sigma_{K^+}) + \mathcal{W}(\Sigma_{K^+}) = \mathbb{R}^3$ ($u_1 = x_2$, $u_2 = -x_{02} - 2x_{01}$ yield $x_2 = -2(p+1)^{-1}x_{01}$, $x_1 = (p+1)^{-1}x_{01}$ and thus $y_{K^+} = 0$), but $\begin{bmatrix}E & A & B\\ 0 & C_{K^+} & D_{K^+}\end{bmatrix}$ is *not* right invertible; note that $K^+ \notin \Gamma_{min}$.

In Examples 4.8 and 4.9 we saw that $\mathcal{V}(\Sigma_K) + \mathcal{W}(\Sigma_K) = \mathbb{R}^n$ if $K \in \Gamma$, Σ_K is as in (3.29), and $E'KE = P$, with P representing the optimal cost for either (2.5) or (2.6). This turns out to be *generally* true for all such $K \in \Gamma$, as a result of which we can generalize a statement in Proposition 1.1.

Theorem 4.10.

Let $K^- \in \Gamma$ be such that $P^- = E'K^-E$, with P^- representing the optimal cost for (LQCP)$^-$. Then $K^- \in \Gamma_{min}$ if and only if rank $\left(\begin{bmatrix}E & [A\ B]\\ 0 & F(K^-)\end{bmatrix}\right) = 1 +$ rank $(F(K^-))$.
Let $K^+ \in \Gamma$ be such that $P^+ = E'K^+E$, with P^+ representing the optimal cost for (LQCP)$^+$. Then $K^+ \in \Gamma_{min}$ if and only if rank $\left(\begin{bmatrix}E & [A\ B]\\ 0 & F(K^+)\end{bmatrix}\right) = 1 +$ rank $(F(K^+))$.

Proof. By (3.19), for all $x_0 \in \mathbb{R}^n$ and all $T \geq 0$,

$$\inf\{{}_0\!\int^T y_{K^-}'(t)y_{K^-}(t)dt \mid u \text{ smooth on } [0, T],\ x \in S^T_{sm}(x_0, u)\} = 0,$$

with $y_{K^-}(t) = C_{K^-}x(t) + D_{K^-}u(t)$, $F(K^-) = \begin{bmatrix}C_{K^-}'\\ D_{K^-}'\end{bmatrix}[C_{K^-}\ D_{K^-}]$, and rank $(F(K^-))$ = rank $([C_{K^-}\ D_{K^-}])$. Consequently, by Proposition 3.2, [19, Appendix] and again Proposition 3.2,

$$\inf\{{}_0\!\int^\infty y_{K^-}'(t)y_{K^-}(t)dt \mid u \in C^m_{sm},\ x \in S(x_0, u) \cap C^n_{sm}\} = 0,$$

and thus, by Theorem 4.4, $\mathcal{V}(\Sigma_{K^-}) + \mathcal{W}(\Sigma_{K^-}) = \mathbb{R}^n$, with Σ_{K^-} as in (3.29). It follows that $K^- \in \Gamma_{min}$ if and only if $\begin{bmatrix}E & A & B\\ 0 & C_{K^-} & D_{K^-}\end{bmatrix}$ is of full row rank, by Theorem 3.14 and Proposition 3.13. Finally, $\begin{bmatrix} E & A & B \\ 0 & C_{K^-} & D_{K^-} \end{bmatrix}$ is of full row rank if and only if rank $\left(\begin{bmatrix} E & [A\ B] \\ 0 & F(K^-) \end{bmatrix}\right) = 1 + \text{rank } (F(K^-))$. The second part of the proof runs similar.

Corollary 4.11.

If $\text{im}(E) = \mathbb{R}^l$, then K^- and K^+ in Theorem 4.10 are in $\Gamma_{\min}$.

Remark 4.12.

If for a certain $K^- \in \Gamma_{\min}$, $E'K^-E = P^-$, with P^- representing the optimal cost for $(LQCP)^-$, then, by Theorem 4.10 and (3.30), rank $\left(\begin{bmatrix} E & [A\ B] \\ 0 & F(K^-) \end{bmatrix}\right) = \kappa$. However, if $K^- \in \Gamma$ is such that $E'K^-E = P^-$ and rank $\left(\begin{bmatrix} E & [A\ B] \\ 0 & F(K^-) \end{bmatrix}\right) = \kappa$, then K^- need *not* be rank minimizing. For example, consider the system

$$p\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}u + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_{01} \\ x_{02} \end{bmatrix},\ y = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}u.$$

Observe that $\ker(D) = 0$; yet, (2.8) does *not* hold. The control $u = 0$ yields $x_1 = 0$ and $x_2 = -x_{01}$ (impulsive) and hence $\mathcal{V} + \mathcal{W} = \mathbb{R}^2$. Thus, $P^- = 0$ (Theorem 4.4), and $P^- = E'K^-E$ with $K^- = 0 \in \Gamma$. It follows that (3.27) $\kappa = 3 = \text{rank } \left(\begin{bmatrix} E & [A\ B] \\ 0 & F(0) \end{bmatrix}\right) = \text{rank } \left(\begin{bmatrix} E & A & B \\ 0 & C & D \end{bmatrix}\right)$; however, by Theorem 4.10, $K^- \notin \Gamma_{\min}$, since $1 + \text{rank } (F(0)) = 4$. Analogously, if $K^+ \in \Gamma$ is such that $E'K^+E = P^+$, with P^+ representing the optimal cost for $(LQCP)^+$, and rank $\left(\begin{bmatrix} E & [A\ B] \\ 0 & F(K^+) \end{bmatrix}\right) = \kappa$, then K^+ need *not* be rank minimizing (note that $P^+ = P^-$ in the above-given example). Therefore we cannot say in Theorem 4.10 that e.g. $K^- \in \Gamma_{\min}$ if and only if rank $\left(\begin{bmatrix} E & [A\ B] \\ 0 & F(K^-) \end{bmatrix}\right) = \kappa$.

In the Examples 4.8 and 4.9 we saw that "suitable" K^- and K^+ in Γ for representation of P^- and P^+ need *not* be rank minimizing; Theorem 4.10 explains us why. For this reason, our main results on the optimal costs for $(LQCP)^-$ and $(LQCP)^+$ are formulated in terms of *all* solutions of the LMI. In particular, the set of solutions of the ARE

$$\Lambda := \{K \in \mathbb{R}^{l \times l} \mid K = K',\ \phi(K) = 0\} \tag{4.10}$$

((3.31)) may be fully *irrelevant* w. r. t. determination of the optimal costs (2.5) - (2.6), even if (2.8) holds, $\ker(D) = 0$ and the LQCPs under consideration are solvable (see Example 4.8). Yet, K^- and K^+ are in Λ in two *special* cases, where $\ker(D) = 0$.

Lemma 4.13.

Let $\ker(D) = 0$. Then $\Gamma = \{K \in \mathbb{R}^{l\times l} | \phi(K) \geq 0\}$, and rank $(F(K)) \geq m$ if $K \in \Gamma$.

We stress that in Lemma 4.13, m may be *unequal* to $\kappa - 1$, and hence we cannot say that $\Gamma_{\min} = \{K \in \Gamma | \text{rank } (F(K)) = m\}$ if $\ker(D) = 0$.

Example 4.14.

The system $p[1\ 0]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = [0\ 1]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + u + [1\ 0]\begin{bmatrix} x_{01} \\ x_{02} \end{bmatrix}$, $y = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}u$ is such that $\kappa - 1 = 3 - 1 = 2$, and $m = 1$. Note that $\ker(D) = 0$, (2.8) is valid, and $\text{im}(E) = \mathbb{R}^{l} = \mathbb{R}$. Hence if $K^- \in \Gamma$ is such that $P^- = E'K^-E$, then $K^- \in \Gamma_{\min}$ by Theorem 4.10. Since $K \in \Gamma$ if and only if $\begin{bmatrix} 1 & K & K+1 \\ K & 1 & 0 \\ K+1 & 0 & 1 \end{bmatrix} \geq 0$, it follows that $\Gamma_{\min} = \{-1, 0\}$, and thus $K^- = 0$ and $P^- = 0$. In fact, also $P^+ = 0$, since $P^+ = E'K^+E$ with $K^+ = 0$. The input $u = -x_1$ is optimal for $(LQCP)^-$ as well as for $(LQCP)^+$, and it has infinitely many associated state trajectories; yet only $x_2 = 0$, $x_1 = (p + 1)^{-1}x_{01}$ is optimal for both problems. Observe that $\Lambda = \emptyset$.

Theorem 4.15.

Assume that $\text{im}(E) = \mathbb{R}^{l}$, $\ker(D) = 0$, and that normal rank $\left(\begin{bmatrix} A - sE & B \\ C & D \end{bmatrix}\right) = l + m$. Let $P^- \geq 0$ be such that, for all $x_0 \in \mathbb{R}^n$, $J^-(x_0) = x_0'P^-x_0$. Then there exists a $K^- \in \Lambda$ such that $P^- = E'K^-E$. In addition, $E'K^-E \geq E'KE$ if $K \in \Lambda$ and $\mathcal{V} \subset \ker(E'KE)$. If, moreover, $P^+ \geq 0$ is such that, for all $x_0 \in \mathbb{R}^n$, $J^+(x_0) = x_0'P^+x_0$, then there exists a $K^+ \in \Lambda$ such that $P^+ = E'K^+E$, and $E'K^+E \geq E'KE$ for every $K \in \Lambda$.

Proof. By Lemma 4.13, rank $(F(K)) \geq m$ if $K \in \Gamma$, and rank $(F(K)) = m$ if and only if $K \in \Lambda$. By Theorem 3.14, then, $\Lambda = \Gamma_{\min}$. Now there exists a $K^- \in \Gamma$ such that $E'K^-E = P^-$ (Theorem 3.11), and it then follows from Corollary 4.11 that $K^- \in \Lambda$. Next, if $K \in \Lambda$ and $\mathcal{V} \subset \ker(E'KE)$, then $E'KE \leq E'K^-E$, by Theorem 4.4. Analogously, if $K^+ \in \Gamma$ is such that $P^+ = E'K^+E$, then $K^+ \in \Lambda$ and if $K \in \Lambda$, then $E'KE \leq E'K^+E$, by Theorem 4.7.

Example 4.16.

Consider the system

$$p[1\ 0]\begin{bmatrix}x_1\\x_2\end{bmatrix} = [0\ 1]\begin{bmatrix}x_1\\x_2\end{bmatrix} + u + [1\ 0]\begin{bmatrix}x_{01}\\x_{02}\end{bmatrix},\ y = [1\ 0]\begin{bmatrix}x_1\\x_2\end{bmatrix} + u.$$

We have $\kappa = 2$ and hence $\kappa - 1 = m = 1$. Also,

$$\Lambda = \{K \mid \begin{bmatrix}1 & K\\K & 0\end{bmatrix} - \begin{bmatrix}(K+1)^2 & 0\\0 & 0\end{bmatrix} = 0\} = \{0\} = \Gamma,$$

and $P^- = P^+ = 0$. Observe that (4.9) is satisfied. For every $[x_{01}, x_{02}]'$ the input $u = - x_1$ is optimal for $(LQCP)^+$ as well as $(LQCP)^-$; a possible optimal state trajectory is $x_2 = 0$, $x_1 = (p + 1)^{-1}x_{01}$ (as in Example 4.14). Note that $x_2 = - x_{01}$ (*impulsive*), $x_1 = 0$ is *another* optimal state trajectory that is associated with $u = - x_1$; (2.8) does *not* hold.

Observe that Theorem 4.15 shows the relevance of solutions of the ARE for possibly *singular* LQCPs subject to a possibly *nonsquare* system. However, in the next case, extensively studied in [17], (2.8) is satisfied and E and A are assumed to be square.

Theorem 4.17.

Assume that $\ker(D) = 0$, $l = n$, $\ker(E) \subset \ker(C)$, and $\mathrm{im}(E) + A[\ker(E)] = \mathbb{R}^l$. If, for all $x_0 \in \mathbb{R}^n$, $J^-(x_0) = x_0'P^-x_0$ with $P^- \geq 0$, then there exists a $K^- \in \Lambda$ such that $P^- = E'K^-E$, and $E'K^-E \geq E'KE$ if $\mathcal{V} \subset \ker(E'KE)$ and $K \in \Lambda$. If, for all $x_0 \in \mathbb{R}^n$, $J^+(x_0) = x_0'P^+x_0$ with $P^+ \geq 0$, then there exists a $K^+ \in \Lambda$ such that $P^+ = E'K^+E$, and $E'K^+E \geq E'KE$ for all $K \in \Lambda$.

Proof. First, consider a general system (3.1), decomposed as in (3.4) and (3.8). Assume, moreover, that $\mathrm{im}([C_3\ D_2]) \subset \mathrm{im}([\bar{C}_2\ D_1])$ (3.5), (3.10). Then we have the following result (proven in the Appendix):

$$\text{If } K = \begin{bmatrix}K_{11} & K_{12}\\K_{12}' & K_{22}\end{bmatrix} \in \Gamma, \text{ then rank } (F(K)) = \text{rank } (\bar{F}_1(K_{11})), \qquad (4.11)$$

see (3.21).

Here, $A_{22} = B_{22}F_{22}$ for some *square* F_{22}, $[A_{22}\ A_{23}]$ is invertible, $C_2 = 0$, $C_3 = 0$. Hence $[\bar{C}_2\ D_1] = [- D_2F_{22}\ D_1]$, with F_{22} *invertible*, and thus $\mathrm{im}([C_3\ D_2]) \subset \mathrm{im}([\bar{C}_2\ D_1])$. Consequently, for every $K \in \Gamma$, rank $(F(K))$ = rank $(\bar{F}_1(K_{11}))$, by the foregoing. Now, let $K^- \in \Gamma$ be such that $P^- = E'K^-E$, with $K^- = \begin{bmatrix}K_{11}^- & K_{12}^-\\(K_{12}^-)' & K_{22}^-\end{bmatrix}$, partitioned in accordance with (3.4), (3.8). Then it is easily established that rank $(\begin{bmatrix}E & [A\ B]\\0 & F(K^-)\end{bmatrix}) = e + (1 - e)$

$+ \text{rank } (\bar{F}_1(K_{11}^-)) = 1 + \text{rank } (F(K^-))$ and hence, by Theorem 4.10, $K^- \in \Gamma_{\min}$. Finally, (2.8) holds, as $\text{rank } (\begin{bmatrix} E & 0 \\ C & D \end{bmatrix}) = 0$, and thus, by Corollary 3.15, $\Lambda = \Gamma_{\min}$. Similarly, if $K^+ \in \Gamma$ is such that $P^+ = E'K^+E$, then $K^+ \in \Gamma_{\min} = \Lambda$. The rest follows from the proof of Theorem 4.15.

Example 4.18.

Consider the system

$$p\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}u + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_{01} \\ x_{02} \\ x_{03} \end{bmatrix},$$

$$y = [0 \ 1 \ 1]\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + u.$$

It is readily checked that $\mathcal{V} = \mathbb{R}^3$, and hence $P^- = 0$ by Theorem 4.4; the control $u = 0$ yields $y = 0$. Let $K = \begin{bmatrix} k_{11} & k_{12} & k_{13} \\ k_{12} & k_{22} & k_{23} \\ k_{13} & k_{23} & k_{33} \end{bmatrix}$. Then $K \in \Lambda \Leftrightarrow 0 =$

$$\begin{bmatrix} 2k_{11} & k_{12} + k_{13} & k_{12} + k_{13} \\ k_{12} + k_{13} & 1 + 2k_{23} & 1 + k_{22} + k_{23} \\ k_{12} + k_{13} & 1 + k_{22} + k_{23} & 1 \end{bmatrix} - \begin{bmatrix} k_{11} \\ k_{12} + 1 \\ 1 \end{bmatrix}[k_{11} \quad k_{12} + 1 \quad 1].$$

By Theorem 4.17, $P^- = E'K^-E$, and $K^- = 0 \in \Lambda$. Also, $P^+ = E'K^+E$ with $K^+ \in \Gamma$ and $E'K^+E \geq E'KE$ for every $K \in \Lambda$. If $K \in \Lambda$ then $k_{11} = 0$ or $k_{11} = 2$; in the latter case $k_{12} = 0$, $k_{13} = 2$, $k_{23} = 0$, $k_{22} = 0$. In the former case k_{12} may be arbitrary and $k_{22} = -\frac{1}{2}k_{12}^2$. As for every k_{12},

$$\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \geq \begin{bmatrix} 0 & k_{12} \\ k_{12} & -\frac{1}{2}k_{12}^2 \end{bmatrix},$$

we find that $P^+ = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

Observe that Theorem 4.15 and Theorem 4.17 reduce to corresponding statements in Proposition 1.1 and Corollary 4.5 if $E = I$.

In this Section we have derived characterizations of the optimal costs for $(LQCP)^-$ and $(LQCP)^+$, and we have seen under which conditions optimal inputs and optimal state trajectories exist. However, we have *not* elaborated on the actual *computation* of these inputs and state trajectories. As this issue may be rather complicated, for instance in the case of nonsquare systems, it will be treated in full detail in a future paper. Nevertheless we will conclude the present paper with a statement on *uniqueness* of optimal controls and optimal state trajectories.

Definition 4.19 [24, Definition 4.10].

The system Σ (3.1) will be called *left invertible in the strong sense* if

$$x_0 = 0,\ y = 0 \Rightarrow u = 0,\ Ex = 0.$$

In [24, Definition 4.1] Σ is called left invertible in the *weak* sense if $x_0 = 0$ and $y = 0$ imply that $u = 0$. Strong and weak left-invertibility coincide if $\det(sE - A) \neq 0$, see also [23].

Proposition 4.20 [24, Corollary 4.15].

Assume that $\begin{bmatrix} E & 0 \\ A & B \\ C & D \end{bmatrix}$ is of full column rank. Then the following statements are equivalent.

i) Σ is left invertible in the strong sense.

ii) If $x_0 = 0$, $y = 0$, then $u = 0$, $x = 0$.

iii) $\begin{bmatrix} A - sE & B \\ C & D \end{bmatrix}$ is left invertible as a rational matrix.

Theorem 4.21.

Let $\begin{bmatrix} E & 0 \\ A & B \\ C & D \end{bmatrix}$ be of full column rank. Consider $(LQCP)^-$, and assume that Σ is output stabilizable. Then for every $x_0 \in \mathbb{R}^n$ there exists exactly one optimal $u \in C^m_{imp}$ *and* exactly one optimal $x \in S(x_0, u) \cap C^n_{imp}$ if and only if Σ is left invertible in the strong sense. If this is the case, then the smooth parts of these unique u and x are of the Bohl type. Consider $(LQCP)^+$, and assume that Σ is state stabilizable and that (4.9) is satisfied. Then for every $x_0 \in \mathbb{R}^n$ there exists exactly one optimal $u \in C^m_{imp}$ *and* exactly one optimal $x \in S(x_0, u) \cap C^n_{imp}$ if and only if Σ is left invertible in the strong sense. If this is the case, then the smooth parts of these unique u and x are of the Bohl type.

Proof. Consider the system Σ, decomposed as in (3.4), (3.8), and the system (3.11) - (3.12). Directly, $\ker(\begin{bmatrix} E & 0 \\ A & B \\ C & D \end{bmatrix}) = 0 \Leftrightarrow \ker(\begin{bmatrix} \bar{A}_{12} & B_{11} \\ \bar{C}_2 & D_1 \end{bmatrix}) = 0$.

Consequently, Σ is left invertible in the strong sense if and only if (3.11) - (3.12) is left invertible (in either sense), by Proposition 3.2 (or by combination of Proposition 4.20 with [10, Theorem 3.26] or [23, Theorem 3.9]). It is known that for every $z_0 \in \mathbb{R}^e$ there exists exactly one optimal $\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \in \mathcal{C}_{imp}^{n+m-l}$ for $(LQCP)^-$ subject to (3.11) - (3.12) if and only if (3.11) - (3.12) is left invertible, and the smooth parts of this optimal input and of the resulting state trajectory are of the Bohl type [10], [6], [8]. Together with Theorem 4.4, this proves the first part. The second half follows analogously, by using Theorem 4.7 instead of Theorem 4.4.

The systems in the examples are all left invertible in the strong sense, except for the system in Example 4.16. Note that one may assume without loss of generality that $\ker(\begin{bmatrix} E & 0 \\ A & B \\ C & D \end{bmatrix}) = 0$ for a system (3.1).

Our final statenent generalizes Corollary 3.15 for strongly left invertible square systems.

Proposition 4.22.

Assume that $l = n$, that $\ker(D) = 0$ and $\ker(\begin{bmatrix} E & 0 \\ A & B \\ C & D \end{bmatrix}) = 0$, and that Σ is left invertible in the strong sense. Then $\Lambda = \Gamma_{min}$.

Proof. By Lemma 4.13, rank $(F(K)) = m$ if and only if $K \in \Lambda$. Since, by Proposition 4.20, $\kappa = l + m$ (3.27), the claim follows from (3.30).

In Example 4.8, $l = n$, $\ker(D) = 0$ and (2.8) holds; the system is left invertible in the strong sense and we saw that $\Gamma_{min} = \Lambda = \emptyset$. The example in Remark 4.12 also satisfies all requirements of Proposition 4.22; yet, (2.8) does *not* hold. It is easily found that

$$\Gamma_{min} = \{K = \begin{bmatrix} k_{11} & k_{12} \\ k_{12} & k_{22} \end{bmatrix} | k_{11} = 0,\ k_{12} = 1 \pm 2^{1/2}\}.$$

Note that l + rank $(F(K)) = 2 + 1 = 3 =$ rank $(\begin{bmatrix} E & [A\ B] \\ 0 & F(K) \end{bmatrix})$ if $K \in \Lambda$, in accordance with Theorem 4.10, amd that, for every $K \in \Gamma_{min}$, $P^+ = P^- = 0 = E'KE$; however, also $0 = E'K_0E$ with $K_0 = 0 \in \Gamma$, $K_0 \notin \Gamma_{min}$. Similar observations can be made for the extended version of **Example 1.3**.

Conclusions.

In this paper we have defined and investigated linear-quadratic control problems (LQCPs) subject to general implicit continuous-time systems. The optimal costs can be interpreted as solutions of the newly defined concept of Dissipation Inequality (DI) for implicit systems. These interpretations have been converted into *unique* representations of the optimal costs by certain solutions of our concept of Linear Matrix Inequality (LMI). In this way the optimal costs are determined by the *original system coefficients* only, and the results are valid for *regular* as well as for *singular* problems. In particular, they reduce to the classical ones if the system is standard. The notion of *rank minimizing* solutions of the LMI, well-known with respect to standard systems, has meaning for implicit systems as well, even if there is no transfer function, and we have derived under which conditions the optimal costs for the LQCPs under consideration can be characterized by rank minimizing solutions of the LMI. In addition, we have shown the possible relevance in this connection of the algebraic Riccati equation for *both* regular *and* singular LQCPs. Further, we have proven that the optimal cost for the LQCP without (with) stability is finite for every inital condition if and only if the underlying system is output (state) stabilizable. Optimal inputs and optimal state trajectories always exist for the former problem, provided that the optimal cost is finite, and they exist for the latter problem if and only if, again, the optimal cost is finite, and the system matrix has no rank deficit on the imaginary axis. Moreover, these inputs and state trajectories are unique if the implicit system is left invertible in the strong sense.

Appendix.

Lemma 1.

Consider the implicit system $pEx = Ax + Bu + Ex_0$, with $\ker(B) = 0$. Then
$I_C = 0 \Leftrightarrow [sE - A,\ - B]$ is left unimodular.

Proof. $\Leftarrow$ Let $x_0 \in I_C$. Then, for some $u \in C_{sm}^m$ and $x \in S(x_0,\ u) \cap C_{sm}^n$, $x(0^+) = x_0$. Since $[sE - A,\ - B]$ has a polynomial left inverse $L(s)$, it follows that $\begin{bmatrix} x \\ u \end{bmatrix} = L(p)Ex_0$ is impulsive, and hence $u = 0$, $x = 0$ and $x_0 = 0$. $\Rightarrow$ If $I_C = 0$, then there are no consistent points in $\mathbf{C}^n$ either. Now, assume that $\lambda \in \mathbf{C}$ and $(\lambda E - A)\bar{x}_0 - B\bar{u}_0 = 0$, $\bar{x}_0 \in \mathbf{C}^n$, $\bar{u}_0 \in \mathbf{C}^m$. Then the input $\bar{u}(t) = e^{\lambda t}\bar{u}_0$ and the state trajectory $\bar{x}(t) = e^{\lambda t}\bar{x}_0$ are both smooth, $\bar{x}(0^+) = \bar{x}_0$, and, for all $t > 0$, $E(d/dt)\bar{x}(t) = \lambda E\bar{x}(t) = A\bar{x}(t) + B\bar{u}(t)$. We conclude that $\bar{x}_0 = 0$. Consequently, $B\bar{u}_0 = 0$ and thus $\bar{u}_0 = 0$; $[\lambda E - A,\ - B]$ is shown to be left invertible for every $\lambda \in \mathbf{C}$.

Proof of Lemma 3.10.

Consider $\bar{F}(K^-)$ (3.24), with $\bar{F}_1(K_{11}^-) \geq 0$, and K_{12}^- chosen such that (3.25) holds. From (3.22) we deduce that

$$(E_{11}'K_{11}^-[\bar{A}_{12}\ B_{11}] + \bar{C}_1'[\bar{C}_2\ D_1]) =$$

$$(E_{11}'K_{11}^-[\bar{A}_{12}\ B_{11}] + \bar{C}_1'[\bar{C}_2\ D_1])\left(\begin{bmatrix} \bar{C}_2' \\ D_1' \end{bmatrix}[\bar{C}_2\ D_1]\right)^+\begin{bmatrix} \bar{C}_2' \\ D_1' \end{bmatrix}[\bar{C}_2\ D_1] \qquad (A.1)$$

and $\bar{C}_1'\bar{C}_1 + \bar{A}_{11}'K_{11}^-E_{11} + E_{11}'K_{11}^-\bar{A}_{11}$ -

$$(E_{11}'K_{11}^-[\bar{A}_{12}\ B_{11}] + \bar{C}_1'[\bar{C}_2\ D_1])\left(\begin{bmatrix} \bar{C}_2' \\ D_1' \end{bmatrix}[\bar{C}_2\ D_1]\right)^+ \times$$

$$\left(\begin{bmatrix} \bar{A}_{12}' \\ B_{11}' \end{bmatrix}K_{11}^-E_{11} + \begin{bmatrix} \bar{C}_2' \\ D_1' \end{bmatrix}\bar{C}_1\right) \geq 0. \qquad (A.2)$$

On the other hand, by (3.25), $\bar{F}(K^-) \geq 0$ if and only if

$$\bar{F}_1(K_{11}^-) - \begin{bmatrix} (E_{11}'K_{11}^-[\bar{A}_{12}\ B_{11}] + \bar{C}_1'[\bar{C}_2\ D_1])\left(\begin{bmatrix} \bar{C}_2' \\ D_1' \end{bmatrix}[\bar{C}_2\ D_1]\right)^+\begin{bmatrix} \bar{C}_2' \\ D_1' \end{bmatrix} \\ \begin{bmatrix} \bar{C}_2' \\ D_1' \end{bmatrix} \end{bmatrix} \times$$

$$[C_3\ D_2]\left(\begin{bmatrix} C_3' \\ D_2' \end{bmatrix}[C_3\ D_2]\right)^+\begin{bmatrix} C_3' \\ D_2' \end{bmatrix} \times$$

$$\left[[\bar{C}_2\ D_1]\left(\begin{bmatrix} \bar{C}_2' \\ D_1' \end{bmatrix}[\bar{C}_2\ D_1]\right)^+\left(\begin{bmatrix} \bar{A}_{12}' \\ B_{11}' \end{bmatrix}K_{11}^-E_{11} + \begin{bmatrix} \bar{C}_2' \\ D_1' \end{bmatrix}\bar{C}_1\right),\ [\bar{C}_2\ D_1]\right] \qquad (A.3)$$

is positive semidefinite.

Set $P_{[C_3\ D_2]} := I - [C_3\ D_2]\left(\begin{bmatrix} C_3' \\ D_2' \end{bmatrix}[C_3\ D_2]\right)^+\begin{bmatrix} C_3' \\ D_2' \end{bmatrix}$. It follows from (A.1) that the right upper block of (A.3) is equal to

$$(E_{11}'K_{11}^-[\bar{A}_{12}\ B_{11}] + \bar{C}_1'[\bar{C}_2\ D_1])\left(\begin{bmatrix} \bar{C}_2' \\ D_1' \end{bmatrix}[\bar{C}_2\ D_1]\right)^+\begin{bmatrix} \bar{C}_2' \\ D_1' \end{bmatrix}P_{[C_3\ D_2]}[\bar{C}_2\ D_1]$$

and the right lower block of (A.3) equals $\begin{bmatrix} \bar{C}_2' \\ D_1' \end{bmatrix}P_{[C_3\ D_2]}[\bar{C}_2\ D_1]$. Finally, the left upper block of (A.3) equals

$$\bar{C}_1'\bar{C}_1 + E_{11}'K_{11}^-\bar{A}_{11} + \bar{A}_{11}'K_{11}^-E_{11} -$$
$$(E_{11}'K_{11}^-[\bar{A}_{12}\ B_{11}] + \bar{C}_1'[\bar{C}_2\ D_1])\left(\begin{bmatrix} \bar{C}_2' \\ D_1' \end{bmatrix}[\bar{C}_2\ D_1]\right)^+\begin{bmatrix} \bar{C}_2' \\ D_1' \end{bmatrix}(I - P_{[C_3\ D_2]})\times$$
$$[\bar{C}_2\ D_1]\left(\begin{bmatrix} \bar{C}_2' \\ D_1' \end{bmatrix}[\bar{C}_2\ D_1]\right)^+\left(\begin{bmatrix} \bar{A}_{12}' \\ B_{11}' \end{bmatrix}K_{11}^-E_{11} + \begin{bmatrix} \bar{C}_2' \\ D_1' \end{bmatrix}\bar{C}_1\right).$$

It follows that the block matrix (A.3) ≥ 0 if and only if (A.2) holds, and this completes the proof.

Proof of Theorem 3.14.

Since (3.3) holds, there exist $N \in \mathbb{R}^{(n+m-1)\times n}$ and $M \in \mathbb{R}^{(n+m-1)\times m}$ such that $\det\left(\begin{bmatrix} sE & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} A & B \\ N & M \end{bmatrix}\right) \neq 0$; take in (3.4) e.g. $N = \begin{bmatrix} 0 & I & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $M = \begin{bmatrix} 0 & 0 \\ I & 0 \end{bmatrix}$. If $\tilde{E} := \begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix}$, $\tilde{A} := \begin{bmatrix} A & B \\ N & M \end{bmatrix}$ and $\tilde{C} := [C\ D]$, then $F(K) = \tilde{C}'\tilde{C} + \tilde{A}'\tilde{K}\tilde{E} + \tilde{E}'\tilde{K}\tilde{A} \geq 0$, with $\tilde{K} := \begin{bmatrix} K & 0 \\ 0 & 0 \end{bmatrix}$. Let $[C_K\ D_K]$, of full row rank, be such that $F(K) = \begin{bmatrix} C_K' \\ D_K' \end{bmatrix}[C_K\ D_K]$. Then rank $(F(K))$ = rank $([C_K\ D_K]) \geq$ normal rank $(\tilde{T}_K(s))$, with $\tilde{T}_K(s) := [C_K\ D_K](s\tilde{E} - \tilde{A})^{-1}\tilde{B}$, where $\tilde{B} := \begin{bmatrix} 0 \\ I_{n+m-1} \end{bmatrix}$.

Since $\tilde{T}_K'(-s)\tilde{T}_K(s) = \tilde{B}'(-s\tilde{E}' - \tilde{A}')^{-1}[\tilde{C}'\tilde{C} + \tilde{A}'\tilde{K}\tilde{E} + \tilde{E}'\tilde{K}\tilde{A})(s\tilde{E} - \tilde{A})^{-1}\tilde{B} =$

$$\tilde{B}'(-s\tilde{E}' - \tilde{A}')^{-1}\tilde{C}'\tilde{C}(s\tilde{E} - \tilde{A})^{-1}\tilde{B} +$$
$$\tilde{B}'(-s\tilde{E}' - \tilde{A}')^{-1}[(\tilde{A}' + s\tilde{E}')\tilde{K}\tilde{E} + \tilde{E}'\tilde{K}(\tilde{A} - s\tilde{E})](s\tilde{E} - \tilde{A})^{-1}\tilde{B} =$$
$$\tilde{B}'(-s\tilde{E}' - \tilde{A}')^{-1}\tilde{C}'\tilde{C}(s\tilde{E} - \tilde{A})^{-1}\tilde{B}$$

($\tilde{E}'\tilde{K}\tilde{B} = 0$), we establish that rank $(F(K))$ = rank $([C_K\ D_K]) \geq$ normal rank $(\tilde{T}(s))$, with $\tilde{T}(s) := \tilde{C}(s\tilde{E} - \tilde{A})^{-1}\tilde{B}$. It is readily checked that

$$\text{normal rank } (\tilde{C}(s\tilde{E} - \tilde{A})^{-1}\tilde{B}) = \text{normal rank }\left(\begin{bmatrix} s\tilde{E} - \tilde{A} & -\tilde{B} \\ \tilde{C} & 0 \end{bmatrix}\right) - (n + m)$$
$$= \text{normal rank }\left(\begin{bmatrix} sE - A & -B & 0 \\ C & D & 0 \\ -N & -M & -I \end{bmatrix}\right) - (n + m)$$
$$= \kappa + (n + m - 1) - (n + m) = \kappa - 1,$$

and, similarly, that normal rank $(\tilde{T}_K(s))$ = normal rank $\left(\begin{bmatrix} sE - A & -B \\ C_K & D_K \end{bmatrix}\right)$ - 1. Consequently, normal rank $\left(\begin{bmatrix} sE - A & -B \\ C_K & D_K \end{bmatrix}\right) = \kappa$, rank $(F(K)) \geq \kappa - 1$ and rank $(F(K)) = \kappa - 1 \Leftrightarrow$

rank $([C_K\ D_K])$ = normal rank $\left(\begin{bmatrix} sE - A & -B \\ C_K & D_K \end{bmatrix}\right) - 1 \Leftrightarrow$

$\begin{bmatrix} sE - A & -B \\ C_K & D_K \end{bmatrix}$ is right invertible as a rational matrix $\Leftrightarrow$

Σ_K is right invertible in the strong sense,

by Proposition 3.13.

Proof of (4.11).

Assume that $K = \begin{bmatrix} K_{11} & K_{12} \\ K_{12}' & K_{22} \end{bmatrix} \in \Gamma$, i.e., $\bar{F}(K) \geq 0$ (3.24), and $\mathrm{im}([C_3\ D_2]) = \mathrm{im}([\bar{C}_2\ D_1])$. Then there exists a real matrix X of appropriate size such that $[C_3\ D_2] = [\bar{C}_2\ D_1]X$, and hence it is easily checked by means of (3.24) that $K \in \Gamma$ if and only if

$\bar{F}_1(K_{11}) \geq 0$, $E_{11}'(K_{11}[A_{13}\ B_{12}] + K_{12}[A_{23}\ B_{22}]) = E_{11}'K_{11}[\bar{A}_{12}\ B_{11}]X$, and rank $(F(K))$ = rank $(\bar{F}_1(K_{11}))$.

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